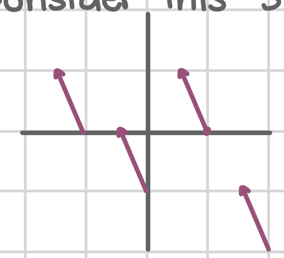
Standard 01: Dot and Cross Product

Review of Vectors

Basic Concepts

Vectors are used to represent quantities that have both a magnitude and a direction. Some examples you may have seen before are force or velocity vectors from physics. For example a force vector might be something like a person pushing a toy car forward with a power of 5 Newtons. Notice that the point of impact does not matter, only the magnitude and the direction.

Consider this sketch of vectors on a 2-D coordinate plane.



While all of these directed line segments originate from a different point, each of these arrows depict the same vector. Particularly, they depict a vector that moves 0.5 units left and 1 unit up. We denote this vector by $\vec{v} = \langle -0.5, 1 \rangle$.
The arrow above the v helps differentiate between a point and a vector, same for the angled brackets.

Notice that we pulled a vector, $\vec{v}$, from a directed line segment $\overrightarrow{AB}$ from the point $A = (x_1, y_1)$ to the point $B = (x_2, y_2)$ by using the equation $\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$. This argument can be extended to 3-D space using the directed line segment $\overrightarrow{AB}$ from $A = (x_1, y_1, z_1)$ to $B = (x_2, y_2, z_2)$ and the equation $\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. Also note that the vector from point A to point B is different from its reverse vector going from point B to point A.
It is customary to draw vectors originating at the origin, these vectors are often called <u>position vectors</u>.

example. Let $A = (2, -7, 0)$ and $B = (1, -3, -5)$. Give the vector for the directed line segments $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

(a) vector described by $\overrightarrow{AB}$

$\vec{v}_1 = \langle 1-2, -3-(-7), -5-0 \rangle$

$= \langle -1, 4, -5 \rangle$

(b) vector described by $\overrightarrow{BA}$

$\vec{v}_2 = \langle 2-1, -7-(-3), 0-(-5) \rangle$

$= \langle 1, -4, 5 \rangle$

Magnitude

We mentioned that a vector consists of two parts: a magnitude and a direction. Sometimes we want to know just the magnitude of the vector a.k.a. the distance of the vector. This formula comes finding the distance traveled by the position vector.

The <u>magnitude</u>, or length, <u>of the vector</u> $\vec{v} = \langle x_1, x_2, x_3 \rangle$ is given by $\|\vec{v}\| = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}$.
This equation can be expanded to a n-dimensional formula for $\vec{v} = \langle x_1, x_2, \dots, x_n \rangle$, $\|\vec{v}\| = \sqrt{(x_1)^2 + (x_2)^2 + \dots + (x_3)^2}$.

example. Determine the magnitude of each of the following vectors,

(a) $\vec{a} = \langle 3, -5, 10 \rangle$

$\|\vec{a}\| = \sqrt{(3)^2 + (-5)^2 + (10)^2}$

$= \sqrt{9 + 25 + 100}$

$= \sqrt{134}$

(b) $\vec{u} = \langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$

$\|\vec{u}\| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{-2}{\sqrt{5}}\right)^2}$

$= \sqrt{\frac{1}{5} + \frac{4}{5}}$

$= 1$

(c) $\vec{w} = \langle 0, 0 \rangle$

$\|\vec{w}\| = \sqrt{(0)^2 + (0)^2}$

$= \sqrt{0 + 0}$

$= 0$

(d) $\vec{i} = \langle 1, 0, 0 \rangle$

$\|\vec{i}\| = \sqrt{(1)^2 + (0)^2 + (0)^2}$

$= \sqrt{1 + 0 + 0}$

$= 1$

Unit Vector

Similarly we may just want the direction the vector points in.

A <u>unit</u> (or direction) <u>vector</u> is a vector of magnitude 1. To find the unit vector divide the vector by its magnitude, $\frac{\vec{v}}{\|\vec{v}\|}$.

example. Find the unit vector for each vector below:

(a) $\vec{a} = \langle 3,-5,10 \rangle$

$\frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\sqrt{134}}\langle 3,-5,10\rangle$

$= \langle \frac{3}{\sqrt{134}}, \frac{-5}{\sqrt{134}}, \frac{10}{\sqrt{134}} \rangle$

(b) $\vec{u} = \langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$

$\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{1}\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$

$= \langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$

(c) $\vec{w} = \langle 0,0 \rangle$

$\frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{0}$!!

no direction

(d) $\vec{i} = \langle 1,0,0 \rangle$

$\frac{\vec{i}}{\|\vec{i}\|} = \frac{1}{1}\langle 1,0,0\rangle$

$= \langle 1,0,0 \rangle$

Special Vectors

The zero vector $\langle 0,0,0\rangle$, often denoted $\vec{0}$, is the vector with no magnitude and direction.

The standard basis vector is a unit vector that moves in the direction of an axis: $\vec{i} = \langle 1,0,0\rangle$, $\vec{j} = \langle 0,1,0\rangle$, $\vec{k} = \langle 0,0,1\rangle$.

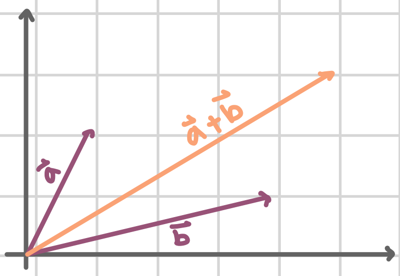
In 2-D space there are only two standard basis vectors, $\vec{i} = \langle 1,0\rangle$ and $\vec{j} = \langle 0,1\rangle$. In n-dimensions, there are n.

Vector Arithmetic

Addition and Subtraction

Given the vectors $\vec{a} = \langle a_1,a_2,a_3\rangle$ and $\vec{b} = \langle b_1,b_2,b_3\rangle$, addition is defined coordinate-wise by the formula:

Note that subtraction is just addition of the negative second vector thus $\vec{a}-\vec{b} = \langle a_1-b_1, a_2-b_2, a_3-b_3\rangle$.



$\vec{a}+\vec{b} = \langle a_1+b_1, a_2+b_2, a_3+b_3\rangle$

$\langle 1,2\rangle + \langle 4,1\rangle = \langle 1+4, 2+1\rangle$

$= \langle 5,3\rangle$

Scalar Multiplication

(Scalar just means a number or one component)

Given a vector $\vec{a} = \langle a_1,a_2,a_3\rangle$ and any number c, the scalar multiplication is $c\vec{a} = \langle ca_1, ca_2, ca_3\rangle$.

Notice that scalar multiplication will stretch (if $c>1$) or shrink (if $c<1$) the original vector but not change the direction.

Standard Basis Vector

We can now see that every vector can be rewritten as multiples and additions of the standard basis vectors, i.e.

$\vec{a} = \langle a_1,a_2,a_3\rangle = \langle a_1,0,0\rangle + \langle 0,a_2,0\rangle + \langle 0,0,a_3\rangle = a_1\langle 1,0,0\rangle + a_2\langle 0,1,0\rangle + a_3\langle 0,0,1\rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

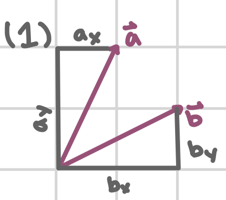
Dot Product

(Scalar Product)

Just like numbers, we want to be able to multiply vectors. We have 2 multiplications for vectors that are important.

The first type of multiplication is the dot product, denoted $\vec{a}\cdot\vec{b}$, and can be found 2 ways:

(1)

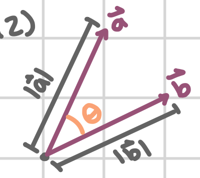


a_x, b_x are the x component of $\vec{a}$ and $\vec{b}$ respectively

a_y, b_y are the y component of $\vec{a}$ and $\vec{b}$ respectively

$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y$

(2)



$|\vec{a}|$ is the magnitude of vector $\vec{a}$

$|\vec{b}|$ is the magnitude of vector $\vec{b}$

θ is the angle between $\vec{a}$ and $\vec{b}$

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$

special properties:

(i) $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$

$\vec{a}\cdot\vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3$

$= a_1^2 + a_2^2 + a_3^2$

$= \left(\sqrt{a_1^2+a_2^2+a_3^2}\right)^2$

$= \|\vec{a}\|^2$

(ii) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$= b_1a_1 + b_2a_2 + b_3a_3$

$= \vec{b}\cdot\vec{a}$

(iii) $\vec{0}\cdot\vec{a} = 0$

$\vec{0}\cdot\vec{a} = 0\cdot a_1 + 0\cdot a_2 + 0\cdot a_3$

$= 0 + 0 + 0$

$= 0$

(iv) $\vec{u} \perp \vec{v}$ if and only if $\vec{u}\cdot\vec{v} = 0$

perpendicular

if $\vec{u}\perp\vec{v}$ then $\theta = \frac{\pi}{2}$, so $\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cos(\pi/2) = 0$

if $\vec{u}\cdot\vec{v} = 0$ then $\|\vec{a}\|\,\|\vec{b}\|\cos\theta = 0$ so $\cos\theta = 0$ ie. $\theta = \pi/2$

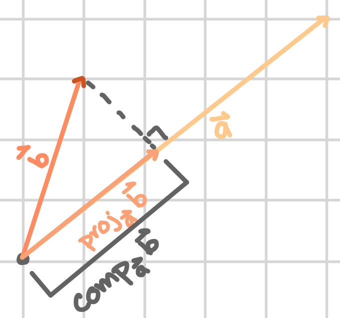
example. Let $\vec{a} = \langle -1,1,2 \rangle$ and $\vec{b} = \langle 0,1,1 \rangle$. Use the formulas to find θ, the angle between $\vec{a}$ and $\vec{b}$.

It is important to note $|\vec{a}| \cdot |\vec{b}| \cdot \cos\theta = \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Thus, $\sqrt{(-1)^2+(1)^2+(2)^2}\sqrt{(0)^2+(1)^2+(1)^2} \cdot \cos\theta = \vec{a}\cdot\vec{b} = (-1)(0)+(1)(1)+(2)(1)$

$$\sqrt{6}\sqrt{2} \cdot \cos\theta = \vec{a}\cdot\vec{b} = 0+1+2$$
$$\sqrt{12} \cdot \cos\theta = \vec{a}\cdot\vec{b} = 3$$
$$2\sqrt{3} \cdot \cos\theta = \vec{a}\cdot\vec{b} = 3$$
$$\cos\theta = \frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$
$$\cos\theta = \frac{3\sqrt{3}}{2\cdot 3}$$
$$\cos\theta = \frac{\sqrt{3}}{2}$$
$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$\theta = \frac{\pi}{6}$$

Component & Projection



It is helpful to see or know what a vector would look like if projected in the direction of another. This is like saying, what is the shadow of one vector on another. We first compute the component of $\vec{b}$ along $\vec{a}$, i.e. how much of $\vec{a}$ should $\vec{b}$ take up and then we multiply this by the direction (or unit vector) of $\vec{a}$. This gives us the 2 parts of a vector, a distance and a direction.

$$\mathrm{comp}_{\vec{a}}\vec{b} = \frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|} = \frac{b_1 \cdot a_1 + b_2 \cdot a_2 + b_3 \cdot a_3}{\sqrt{a_1^2+a_2^2+a_3^2}} \qquad \mathrm{proj}_{\vec{a}}\vec{b} = \mathrm{comp}_{\vec{a}}\vec{b} \cdot \frac{\vec{a}}{\|\vec{a}\|} = \frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|} \cdot \frac{\vec{a}}{\|\vec{a}\|}$$

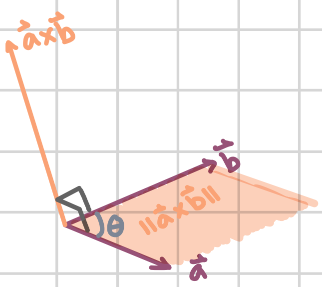
example. Let $\vec{b} = \langle 0,1,1 \rangle$ and $\vec{a} = \langle -1,1,2 \rangle$. Compute the scalar component of $\vec{b}$ along $\vec{a}$ and the vector projecture of $\vec{b}$ along $\vec{a}$.

$$\mathrm{comp}_{\vec{a}}\vec{b} = \frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|} = \frac{\langle 0,1,1\rangle \cdot \langle -1,1,2\rangle}{\|\langle -1,1,2\rangle\|} = \frac{0\cdot(-1)+1\cdot 1+1\cdot 2}{\sqrt{(-1)^2+(1)^2+2^2}} = 3/\sqrt{6}$$

$$\mathrm{proj}_{\vec{a}}\vec{b} = \mathrm{comp}_{\vec{a}}\vec{b} \cdot \frac{\vec{a}}{\|\vec{a}\|} = 3/\sqrt{6} \cdot \frac{\langle -1,1,2\rangle}{\sqrt{6}} = 3/\sqrt{36} \cdot \langle -1,1,2\rangle = 3/6 \cdot \langle -1,1,2\rangle = \langle -\tfrac{1}{2}, \tfrac{1}{2}, 1\rangle$$

Cross Product (Vector Product)

The second type of multiplication gives us a vector. The cross product, denoted $\vec{a}\times\vec{b}$, gives a new vector that is perpendicular:



the magnitude of the cross product is the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$

- the length is 0 when the vectors $\vec{a}$ and $\vec{b}$ point in the same or opposite direction
- the length is at maximum when $\vec{a}$ and $\vec{b}$ are at right angles

$$\vec{a}\times\vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$= \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \vec{i}(a_2 b_3 - a_3 b_2) - \vec{j}(a_1 b_3 - a_3 b_1) + \vec{k}(a_1 b_2 - a_2 b_1)$$

Alternative way to compute cross product:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\begin{matrix} \vec{i} & \vec{j} \\ a_1 & a_2 \\ b_1 & b_2 \end{matrix} \qquad \begin{aligned} &= a_2 b_3 \vec{i} + a_3 b_1 \vec{j} + a_1 b_2 \vec{k} - a_2 b_1 \vec{k} - a_3 b_2 \vec{i} - a_1 b_3 \vec{j} \\ &= (a_2 b_3 - a_3 b_2)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k} \end{aligned}$$

straight bars denote determinate
square brackets denote a matrix

A more geometric approach gives the formula, $\|\vec{a}\times\vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \sin\theta$ for $0 \le \theta \le \pi$.

example. Let $\vec{a} = \langle -1, 1, 2 \rangle$ and $\vec{b} = \langle 0, 1, 1 \rangle$. Calculate $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \overset{+}{1} & \overset{-}{2} \\ 1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} \overset{+}{-1} & \overset{-}{2} \\ 0 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} \overset{+}{-1} & \overset{-}{1} \\ 0 & 1 \end{vmatrix} \vec{k}$$

$$= (1 \cdot 1 - 2 \cdot 1)\vec{i} - (-1 \cdot 1 - 2 \cdot 0)\vec{j} + (-1 \cdot 1 - 1 \cdot 0)\vec{k}$$

$$= (1-2)\vec{i} - (-1+0)\vec{j} + (-1+0)\vec{k}$$

$$= -\vec{i} + \vec{j} - \vec{k}$$

$$= \langle -1, 1, -1 \rangle$$

$$\vec{a} \times \vec{b} = \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{matrix} \right| \begin{matrix} \vec{i} & \vec{j} \\ -1 & 1 \\ 0 & 1 \end{matrix}$$

$$= (1)(1)\vec{i} + (2)(0)\vec{j} + (-1)(1)\vec{k} - (1)(0)\vec{k} - (2)(1)\vec{i} - (-1)(1)\vec{j}$$

$$= 1\vec{i} + 0\vec{j} - 1\vec{k} - 0\vec{k} - 2\vec{i} + 1\vec{j}$$

$$= (1-2)\vec{i} + (0+1)\vec{j} \; (-1-0)\vec{k}$$

$$= (-1)\vec{i} + (1)\vec{j} + (-1)\vec{k}.$$

example. Using the information you know about cross product, find $\|\vec{u} \times \vec{v}\|$ given $\|\vec{u}\| = 3$, $\|\vec{v}\| = 6$, and $\vec{u} \cdot \vec{v} = -9$.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

$$-9 = (3)(6)\cos\theta$$

$$-1/2 = \cos\theta$$

$$\theta = \frac{2\pi}{3}$$

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \cdot \|\vec{v}\| \sin\theta$$

$$\|\vec{u} \times \vec{v}\| = (3) \cdot (6) \cdot \sin\left(\frac{2\pi}{3}\right)$$

$$\|\vec{u} \times \vec{v}\| = 18 \cdot \sqrt{3}/2$$

$$\|\vec{u} \times \vec{v}\| = 9\sqrt{3}$$