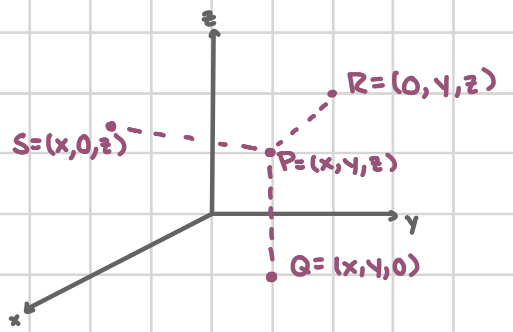
Standard 02: Lines

Introduction to the 3-D Coordinate System

The 3-D coordinate system is often denoted by $\mathbb{R}^3$, mimicking $\mathbb{R}^2$ for the 2-D coordinate system and $\mathbb{R}$ for the 1-D coordinate system. We can also bring this out to a n-dimensional coordinate system denoted by $\mathbb{R}^n$. Visually the 3-D coordinate system is shown below:

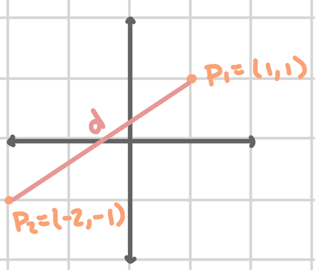


This is the standard placement of the axes with it assumed that only the positive directions are shown. We will add the negative axes only if needed and label them. The point $P=(x,y,z)$ is a general point sitting in 3-D space. We may use the word projection to describe going from the xyz-system to any of the 2-D planes, e.g. if you drop down to $z=0$ then we get the point $Q=(x,y,0)$ in the xy-plane. In addition, we can find points S and R in the xz and yz plane, respectively.

Properties of 3-D

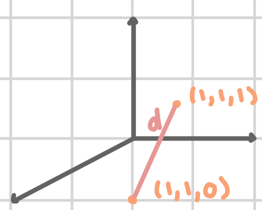
Many of the formulas you are familiar with in $\mathbb{R}^2$ have natural extensions into the $\mathbb{R}^3$ coordinate system. For example, the distance between two points:

2-D Space



$$
\begin{aligned}
d &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
&= \sqrt{(1-(-2))^2+(1-(-1))^2} \\
&= \sqrt{(3)^2+(2)^2} \\
&= \sqrt{9+4} \\
&= \sqrt{15}
\end{aligned}
$$

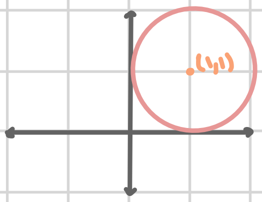
3-D Space



$$
\begin{aligned}
d &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \\
&= \sqrt{(1-1)^2+(1-1)^2+(1-0)^2} \\
&= \sqrt{(0)^2+(0)^2+(1)^2} \\
&= \sqrt{1} \\
&= 1
\end{aligned}
$$

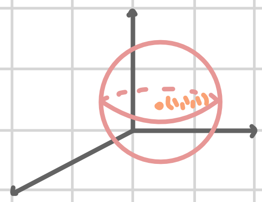
Likewise, the general equation of a circle with center (h,k) and radius r extends to a sphere with center (h,k,ℓ) and radius r:

2-D Space



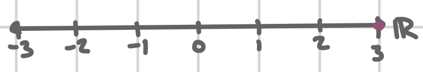
$$(x-h)^2+(y-k)^2=r^2$$
$$(x-1)^2+(y-1)^2=(1)^2$$

3-D Space

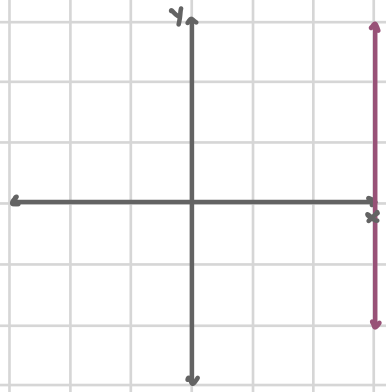


$$(x-h)^2+(y-k)^2+(z-\ell)^2=r^2$$
$$(x-1)^2+(y-1)^2+(z-1)^2=(1)^2$$

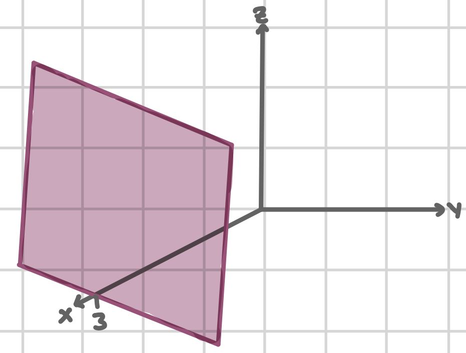
Not everything about $\mathbb{R}^2$ translates to $\mathbb{R}^3$ the way we expect. For example, let's graph $x=3$ in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$.



$x=3$ is a point on $\mathbb{R}$



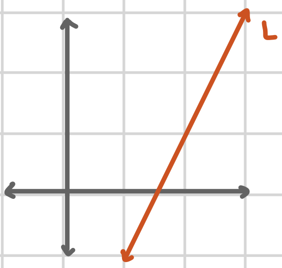
$x=3$ is a line in $\mathbb{R}^2$ of all points of the form $(3,y)$



$x=3$ is a plane in $\mathbb{R}^3$ that contains all point $(3,y,z)$

Lines

Lines in 2D

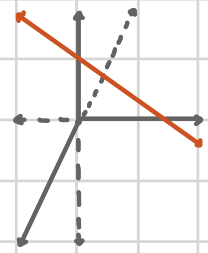


A line in 2D space requires two things:

- a y-intercept, b
- a slope, m

slope intercept form $y = mx + b$

Lines in 3D



A line in 3D space requires two things:

- a point on the line, $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$
- a direction, $\vec{v} = \langle a, b, c \rangle$

line equation: $\vec{r} = \vec{r}_0 + t \cdot \vec{v}$

example. Find the equation of the line passing through $(1,2,3)$ and parallel to $\langle 1,3,-2 \rangle$.

We are given a point on the line and the direction of the line is the same as the direction of $\langle 1,3,-2 \rangle$. Thus our line is of the form $\vec{r} = (1,2,3) + t \cdot \frac{\langle 1,3,-2 \rangle}{|\langle 1,3,-2 \rangle|}$. Note that t is any real number so we can redefine it to be $t_1 = \frac{t}{|\langle 1,3,-2 \rangle|}$ and simplify the equation to $\vec{r} = (1,2,3) + t_1 \langle 1,3,-2 \rangle$.

Forms of a Line

1. Vector Form

The form discussed above, $\vec{r} = \vec{r}_0 + t \cdot \vec{v} = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$, is called the vector form of the equation of a line.

2. Parametric Form

This is a slight rewrite of vector form into component: $\langle x, y, z \rangle = \vec{r} = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$ becomes a set of equations $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$. This set of equations is called the parametric form of the equation of a line.

3. Symmetric Form

Each of the equations in parametric form have a t value, allowing us to solve each for t and set them equal for our final form: symmetric equations of the line. Note that a, b, c must be nonzero in this form.

$t = \frac{x - x_0}{a}$, $t = \frac{y - y_0}{b}$, $t = \frac{z - z_0}{c}$ thus $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$.

example. Write down the equation of the line that passes through the points $(1,3,-2)$ and $(2,7,-5)$ in all three forms.

We are given two points, but need one point and a direction. One of the ways to find a direction vector is to find the vector represented by going from one point on the line to another point on the line, e.g. $\vec{AB}$ for $A = (1,3,-2)$ and $B = (2,7,-5)$. So, $\vec{v} = \vec{AB} = \langle 2-1, 7-3, -5-(-2) \rangle = \langle 1, 4, -3 \rangle$. We can now use the vector form equation to solve for one form.

(a) vector form
$\vec{r} = (1,3,-2) + t \langle 1,4,-3 \rangle$

(b) parametric form
$x = 1 + 1t$, $y = 3 + 4t$, $z = -2 - 3t$

(c) symmetric form
$\frac{x-1}{1} = \frac{y-3}{4} = \frac{z+2}{-3}$

example. Does the point $(0,0,2)$ belong to the line $\vec{r}(t) = \langle 1,2,3 \rangle + t \langle 1,-2,1 \rangle$?

The point $(0,0,2)$ is on the line if and only if there exists a single t such that $0 = 1 + t$, $0 = 2 - 2t$, $2 = 3 + t$
The t values are not equivalent so this point is not on the line. $t = -1$, $t = 1$, $t = 2/3$

example. Determine if the line that passes through the point $(2,-1,3)$ and is parallel to the line given by $x = 1 + t$, $y = 3 + 4t$, $z = -2 - 3t$ passes through the xz-plane. If it does give the coordinates of that point.

We are directly given a point and sneakily given a direction through a parallel line. We extract the direction vector of the parallel line by finding $\vec{v} = \langle a, b, c \rangle$ from our system. The a, b, c for parametric form are the constants in front of the t variable for each equation. So $\vec{v} = \langle 1, 4, -3 \rangle$. The equation of our line can now be written it parametric form using our given point: $x = 2 + t$, $y = -1 + 4t$, $z = 3 - 3t$. The xz-plane can be described by all points (x,y,z) where $y = 0$. This means our line will pass through the xz-plane only when our $y = -1 + 4t$ equation is zero. We now solve for such a t:

$$y = -1 + 4t = 0$$
$$4t = 1$$
$$t = 1/4$$

So our line passes through the xz-plane when $t = 1/4$ which gives the point $(2 + 1/4, 0, 3 - 3/4) = (9/4, 0, 9/4)$.