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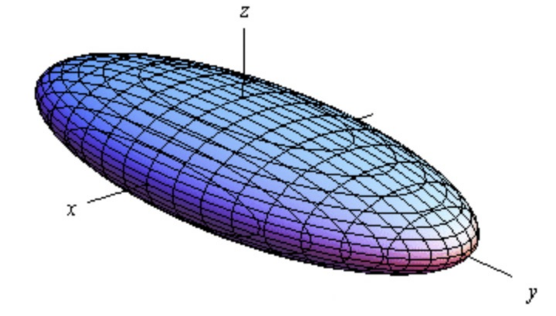
Common Quadric Surfaces

The previous two standards focused on lines and planes, today we want to consider surfaces that commonly appear. In particular we will focus on quadric surfaces, ie. graphs of any equation that can be put into the general form $Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$ where $A, \dots, J$ are constants.

ellipsoid

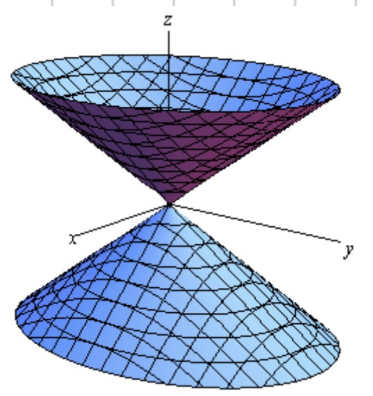
The general equation of an ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.



Notice that the equation above gives an ellipsoid centered at the origin. This may not always be the case but it is easier for note taking purposes. Also, make note that if $a=b=c$ then we have a sphere.

cone

The general equation of a cone is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$.



While this does not look like what we expect a cone to be, we can restrict to $z\geq 0$ or $z\leq 0$ to see the classic cone shape. Alternatively, you can solve the equation for z:

$c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=z^2$

$\frac{c^2}{a^2}x^2+\frac{c^2}{b^2}y^2=z^2$

$\pm\sqrt{\frac{c^2}{a^2}x^2+\frac{c^2}{b^2}y^2}=z$

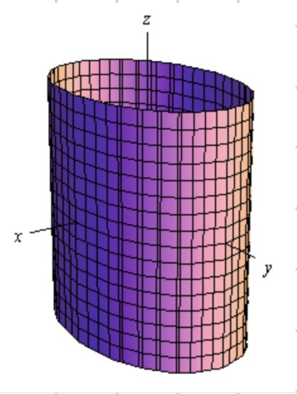
square roots alway return positive numbers so:

$z=\sqrt{A^2x^2+B^2y^2}$ = upper cone $\quad z=-\sqrt{A^2x^2+B^2y^2}$ = lower cone

Furthermore, this cone centers on the z-axis. The variable that sits alone on one side the equal sign will determine the axis the cone "centers" around. For example, to have the cone centered on the x-axis we use the equation $\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{x^2}{a^2}$.

cylinder

The general equation of a cylinder is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

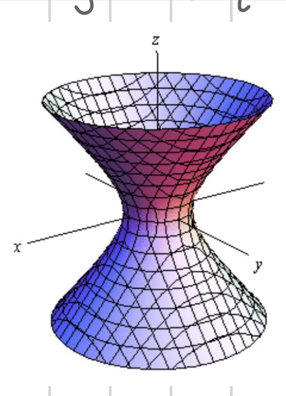


Much like the cone, the cylinder is centered on one axis. The axis correlates to the missing variable. The cross section of this cylinder is a ellipse unless $a=b$, in which case it will be a circle.

We mostly deal with circular cylinders and instead use a simplified equation: $x^2+y^2=r^2$ where r is the radius of the circle cross section.

hyperboloid of one sheet

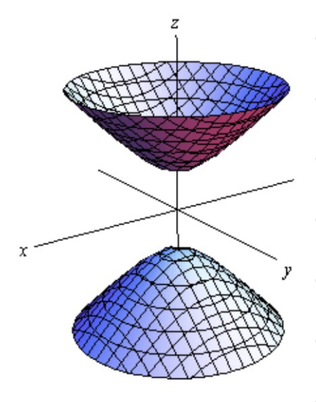
The general equation of a hyperboloid of one sheet is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.



The variable following the negative sign is the axis the hyperboloid centers around.

hyperboloid of two sheets

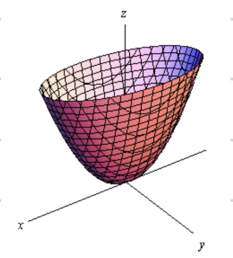
The general equation of a hyperboloid of two sheets is $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.



The variable with the positive sign in front of it will give the axis it is centered on. Notice that the only difference between the one fold and two fold equations is the signs, they are exactly the opposite.

elliptic paraboloid

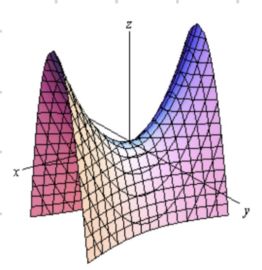
The general equation of an elliptic paraboloid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$.



The equation above has elliptic cross section and if a=b it will have a circle cross section. The variable that isn't squared is the axis that the paraboloid centers around. The sign of the constant c tells us if the paraboloid opens up (positive) and down (negative).

hyperbolic paraboloid

The general equation of a hyperbolic paraboloid is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$.



This should look similar to a saddle. The graph uses a positive c, a negative c would reverse the direction the graph "opens up." Adding or subtracting a constant on the left side will shift the surface up or down.

Domain of Multivariable Functions

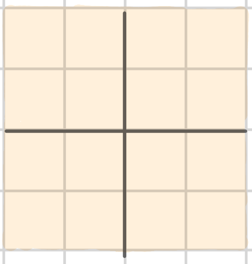
Recall from high school algebra that the domain of a function of a single variable, $y = f(x)$, consisted of all the values of x that we can plug into the function to receive a real number. In this case the domain of the function is an interval (or intervals) of values from the number line. The domain of a function of two variables, $z = f(x,y)$, are regions from two dimensional space and consist of all the coordinate pairs, (x,y), that we can plug into the function to receive a real number.

example. Determine the domain of each of the following:

(a) $f(x,y) = x^2 + y^2 + 6$

$\{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$

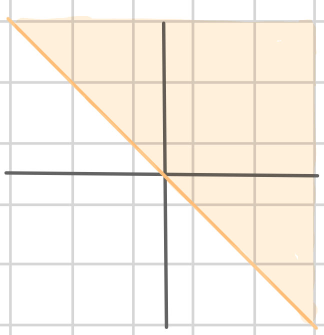
there are no restrictions



(b) $f(x,y) = \sqrt{x+y}$

You can not take the square root of a negative number

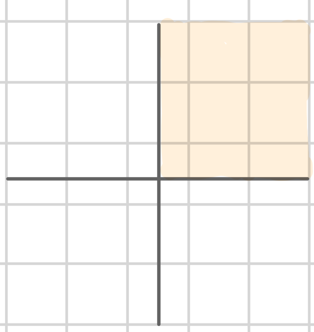
$\{(x,y) \mid x+y \geq 0\}$



(c) $f(x,y) = \sqrt{x} + \sqrt{y}$

You can not take the square root of a negative number

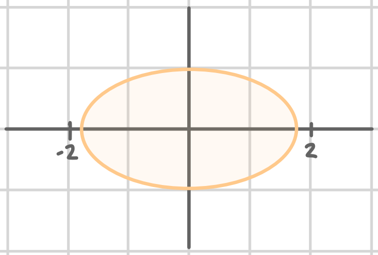
$\{(x,y) \mid x \geq 0, y \geq 0\}$



(d) $f(x,y) = \ln(x^2 + y^2 - 3)$

You can not take the log of a negative or of zero

$\{(x,y) \mid x^2 + y^2 - 3 > 0\}$



Similarly, a function of three variables, $w=f(x,y,z)$, will have a region in three dimensional space as a domain.

example. Determine the domain of $f(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2-9}}$.

We can not take a square root of a negative number and we can not divide by zero:

$x^2+y^2+z^2-9>0$

$x^2+y^2+z^2>9$ which are the points outside of the sphere of radius 3 centered at the origin.

Parameterization

We have already seen our first example of vector-valued functions when we handled equations of lines. The equation of a line, in vector and parametric form, took in a value t and spit out a position vector $\vec{v}$. We aim to do that here with more than just lines.

vector-valued functions

A vector-valued function takes in one or more variables and returns a vector. We will mostly see single variable vector-valued functions, but there are cases we will deal with more. A vector-valued function of a single variable in $\mathbb{R}^2$ and $\mathbb{R}^3$ have the form, $\vec{r}(t)=\langle f(t),g(t)\rangle$ and $\vec{r}(t)=\langle f(t),g(t),h(t)\rangle$, respectively, where $f(t),g(t),h(t)$ are called component functions.

Our goal in this section is to identify vector-valued functions, graph them, and identify the graph given the vector-valued function. We start by identifying their domains, i.e. the set of all t's that give a real value when plugged into all component functions.

example. Determine the domain of the vector-valued function $\vec{r}(t)=\langle\cos(t),\ln(4-t),\sqrt{t+1}\rangle$

We first solve for the domain of each component function and then solve for the t that satisfy all domains.

$$\left.\begin{array}{l}\text{domain of } \cos(t):\ \text{all real numbers}\\ \text{domain of } \ln(4-t):\ t<4\\ \text{domain of } \sqrt{t+1}:\ t\geq -1\end{array}\right\}\ -1\leq t<4$$

Let us start with graphs of 2-dimensional vector valued functions. To sketch the graph all we need to do is plug in some values of t that fit the domain and plot points that correspond to the resulting position vector.

example. Sketch the graph of each of the following vector functions.

(a) $\vec{r}(t)=\langle t^3,3\rangle$

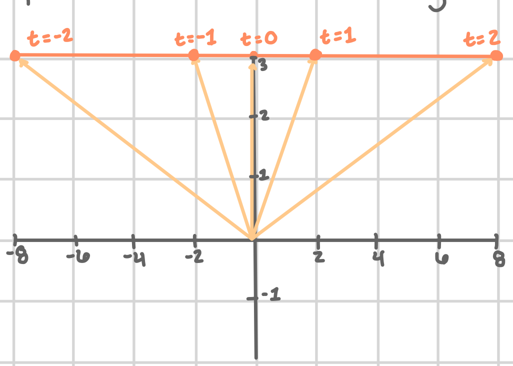
$t=-2$: $\vec{r}(-2)=\langle -8,3\rangle$

$t=-1$: $\vec{r}(-1)=\langle -1,3\rangle$

$t=0$: $\vec{r}(0)=\langle 0,3\rangle$

$t=1$: $\vec{r}(1)=\langle 1,3\rangle$

$t=2$: $\vec{r}(2)=\langle 8,3\rangle$



(b) $\vec{r}(t)=\langle t,t^3-2t+1\rangle$

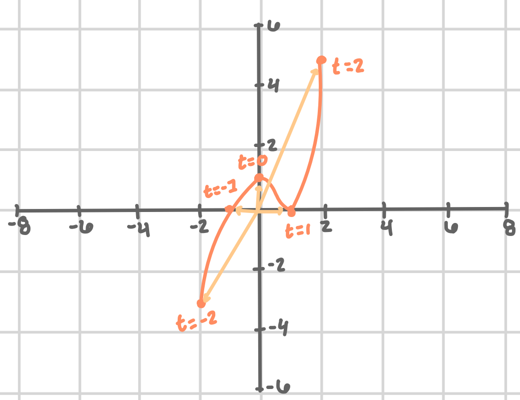
$t=-2$: $\vec{r}(-2)=\langle -2,-3\rangle$

$t=-1$: $\vec{r}(-1)=\langle -1,0\rangle$

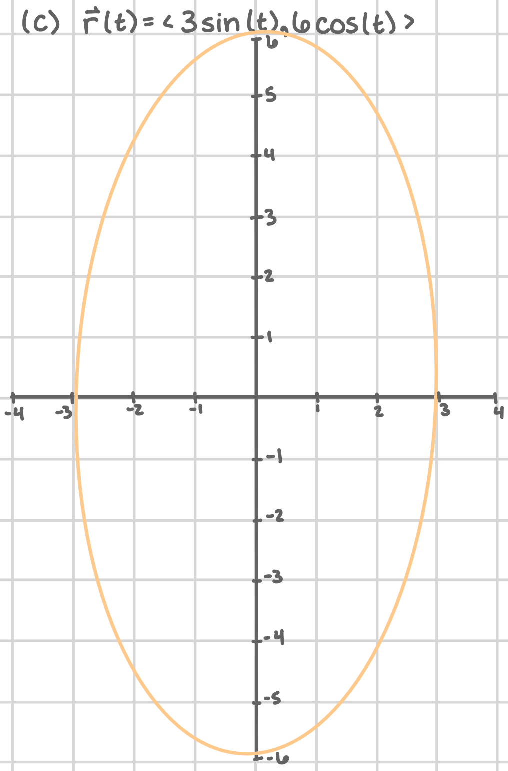
$t=0$: $\vec{r}(0)=\langle 0,1\rangle$

$t=1$: $\vec{r}(1)=\langle 1,0\rangle$

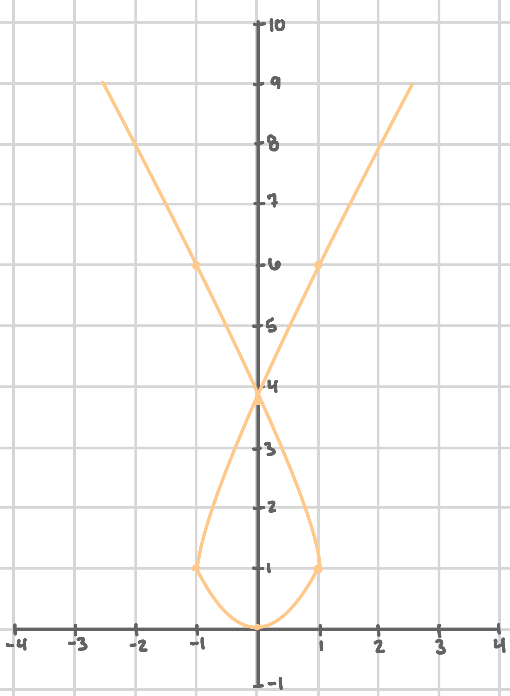
$t=2$: $\vec{r}(2)=\langle 2,5\rangle$



(c) $\vec{r}(t)=\langle 3\sin(t),6\cos(t)\rangle$



(d) $\vec{r}(t)=\langle t-2\sin(t),t^2\rangle$



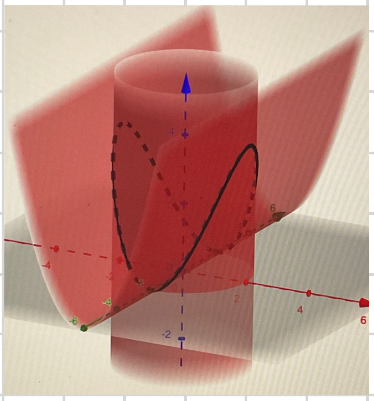
parametric equations

Notice that our third example graphed an ellipse. If we break the vector-valued function into the component functions, $x=6\cos(t)$ and $y=3\sin(t)$, then we can more clearly see why an ellipse appears. This may have even been covered in calculus II when going over parametric equations, it is one set of parametric equations that gives an ellipse. Any vector-valued function can be broken down into the parametric equations $x=f(t)$, $y=g(t)$, $z=h(t)$.

intersection of surfaces

Given two surfaces, it is common to ask for their intersection if one exists. We have already seen an example of this in standard 03: planes when asked to find the line of intersection between two planes. In this section we have seen examples of surfaces that have a non-zero curvature. Imagine what happens to two intersecting planes if you "bend" them to be a paraboloid and a cylinder. Naturally, as you "bend" the surface, the intersection become "bent". We call these "bent" lines curves and often parameterize them for ease of reading.

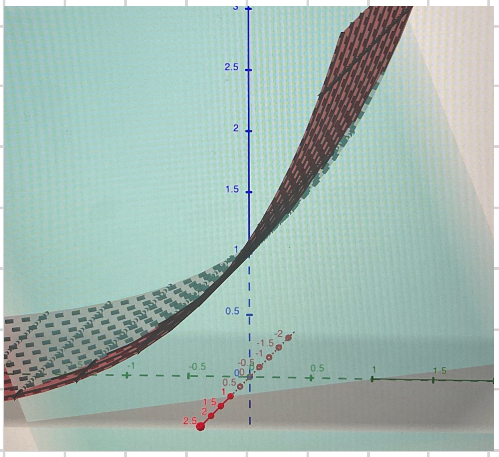
example. Find the curve of intersection for the two surfaces $z=x^2$ and $x^2+y^2=4$.



The curve of intersection is the curve, $\langle x(t), y(t), z(t)\rangle$, that satisfies both equations.
Note that the first equation, $z=x^2$, gives us a parameterization of z using x's parameterization. It is left to find a parameterization for x and y. The function $x^2+y^2=4$ outlines a circle of radius 2, therefore we can use the parameterization $x(t)=a\sin(t)$ and $y(t)=a\cos(t)$ where a come from the base equation $(a\sin(t))^2+(a\cos(t))^2=a^2$. Thus $x(t)=2\sin(t)$, $y(t)=2\cos(t)$, and $z(t)=x^2=(2\sin(t))^2$.

Sometimes we aren't so lucky to recieve one variable equal to an expression of another or an equation we know the parameterization for already. Sometimes we have to make our own luck by solving an equation for one variable and choosing the "innermost" variable to be our parameter t.

example. Find the curves of intersection for the two surfaces $x+y+z=1$ and $z=e^y$



Pick $y(t)=t$ as it is the "innermost" variable: $y(t)=t$.
Then we can solve for z: $z(t)=e^{y(t)}=e^t$.
Solve $x+y+z=1$ for the final variable x: $x=1-y-z$.
Plug in known variables: $x(t)=1-y(t)-z(t)=1-t-e^t$

$\langle 1-t-e^t, t, e^t\rangle$