Standard 05: Calculus of Curves

Calculus of Curves

In single-variable calculus, you learnt how to take the limit, derivative, and integral of single variable functions, $f(x)$. This section aims to extend these ideas to vector-valued function, $\vec{r} = \langle x(t), y(t), z(t) \rangle$. These ideas can be extended to $\mathbb{R}^n$.

limits

The limit of a vector-valued function is intuitive: $\lim_{t\to a} \vec{r}(t) = \lim_{t\to a} \langle x(t), y(t), z(t) \rangle = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t) \rangle$. You simply take the limit of each component in the parameterization.

We can extend the definition of continuity of a single variable function to a definition of continuity for vector-valued functions: a vector-valued function $\vec{r}(t)$ is continuous at a if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$.

example. Is the vector-valued function $\vec{r}(t) = \langle t\cos(t), t, t\sin(t) \rangle$ continuous at $t = \pi$?

$\lim_{t\to\pi} \vec{r}(t) = \langle \lim_{t\to\pi} t\cos(t), \lim_{t\to\pi} t, \lim_{t\to\pi} t\sin(t) \rangle$
$= \langle \pi \cdot -1, \pi, \pi \cdot 0 \rangle$
$= \langle -\pi, \pi, 0 \rangle$

$\vec{r}(\pi) = \langle \pi\cos(\pi), \pi, \pi\sin(\pi) \rangle$
$= \langle \pi \cdot -1, \pi, \pi \cdot 0 \rangle$
$= \langle -\pi, \pi, 0 \rangle$

$\vec{r}(t)$ is continuous at π.

derivatives

One of the main applications of limits from single-variable calculus is derivatives. We can extend the single-variable definition the same way we did for limits. The derivative for a vector-valued function, $\vec{v}(t)$, is defined as follows:

$$\frac{d}{dt}(\vec{r}(t)) = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \left\langle \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h}, \lim_{h\to 0} \frac{z(t+h) - z(t)}{h} \right\rangle = \langle x'(t), y'(t), z'(t) \rangle = \left\langle \frac{d}{dt}x(t), \frac{d}{dt}y(t), \frac{d}{dt}z(t) \right\rangle.$$

example. Find the derivative of $\vec{r}(t) = \langle t\cos(t), t, t\sin(t) \rangle$.

$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$x(t) = t\cos(t)$ $\quad x'(t) = 1 \cdot \cos(t) - \sin(t) \cdot 1$

$y(t) = t$ $\quad y'(t) = 1$

$z(t) = t\sin(t)$ $\quad z'(t) = 1 \cdot \sin(t) + \cos(t) \cdot 1$

$\vec{r}\,'(t) = \langle \cos(t) - \sin(t), 1, \sin(t) + \cos(t) \rangle$

properties:

$(\vec{u} + \vec{v})' = \vec{u}\,' + \vec{v}\,'$

$(c\vec{u})' = c \cdot \vec{u}\,'$

$(\vec{u}(\vec{v}(t)))' = \vec{u}\,'(\vec{v}(t)) \cdot \vec{v}\,'(t)$

$(\vec{u} \cdot \vec{v})' = \vec{u}\,'\vec{v} + \vec{u}\,\vec{v}\,'$

$(\vec{u} \bullet \vec{v})' = \vec{u}\,' \bullet \vec{v} + \vec{u} \bullet \vec{v}\,'$

$(\vec{u} \times \vec{v})' = \vec{u}\,' \times \vec{v} + \vec{u} \times \vec{v}\,'$

integration

Lastly, we extend the definition of integrals to vector-valued functions.

- indefinite integral for vector-valued function: $\int \vec{r}(t)\,dt = \langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \rangle$ each component will have a $+C$, use $\vec{C} = \langle C_1, C_2, C_3 \rangle$ to simplify
- definite integral for vector-valued function: $\int_a^b \vec{r}(t)\,dt = \langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \rangle$

example. Compute the following integrals using $\vec{r}(t) = \langle t\sin(t^2), t, \cos(t) \rangle$

(i) $\int \vec{r}(t)\,dt$

$\int \vec{r}(t)\,dt = \langle \int t\sin(t^2)\,dt, \int t\,dt, \int \cos(t)\,dt \rangle$
$= \langle -\frac{1}{2}\cos(t^2) + C_1, \frac{1}{2}t^2 + C_2, \sin(t) + C_3 \rangle$
$= \langle -\frac{1}{2}\cos(t^2), \frac{1}{2}t^2, \sin(t) \rangle + \vec{C}$

(ii) $\int_0^\pi \vec{r}(t)\,dt$

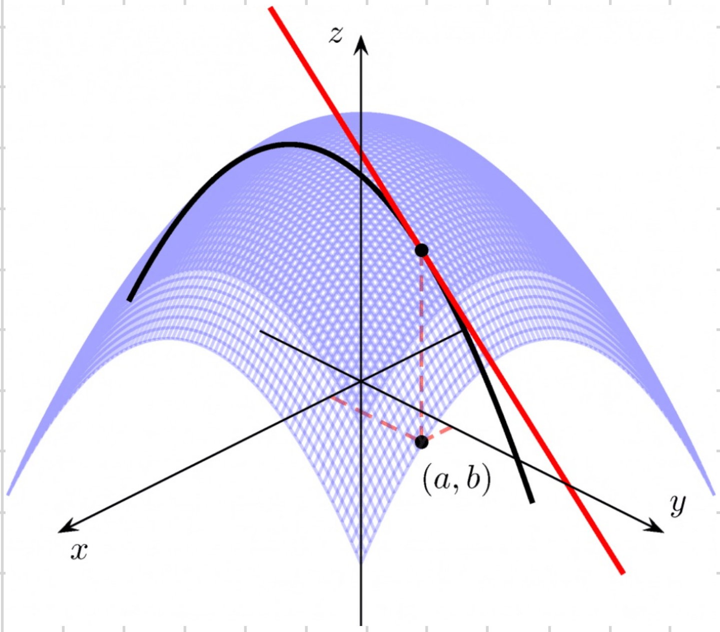
$\int_0^\pi \vec{r}(t)\,dt = \langle \int_0^\pi t\sin(t^2), \int_0^\pi t\,dt, \int_0^\pi \cos(t)\,dt \rangle$
$= \langle -\frac{1}{2}\cos(t^2)\big]_0^\pi, \frac{1}{2}t^2\big]_0^\pi, \sin(t)\big]_0^\pi \rangle$
$= \langle -\frac{1}{2}(\cos(\pi^2) - 1), \frac{1}{2}(\pi^2), 0 \rangle$

Applications

Derivatives and integrals have a lot of applications, in fact the next standard is dedicated to one application of the derivative, we will discuss three short applications in this section.

equation of the tangent line

Recall from Calculus I that the derivative of a function is the slope of the tangent line. For vector-valued functions, the derivative gives a tangent vector that points in the direction of increasing t-values. This vector is used as a direction vector for the tangent.



Given the vector-valued function, $\vec{r}(t)$, we call $\vec{r}\,'(t)$ the tangent vector provided it exists and is not $\vec{0}$. The tangent line to $\vec{r}(t)$ at the point P is then the line that passes through the point P and is parallel to the tangent vector. If $\vec{r}\,'(t)=\vec{0}$ we would have a vector with no magnitude and no direction.

Given that $\vec{r}\,'(t) \neq \vec{0}$, the unit tangent vector to the curve is given by $\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$.

$\vec{r}(t)$ shown in black, tangent line shown in red i.e. $v(t) = \vec{P} + t\,\vec{r}\,'(t)$

example. Find the general formula for the unit tangent vector and the vector equation of the tangent line to the curve given by $\vec{r}(t) = \langle \cos(t), -\sin(t), t \rangle$ at $t = \pi$.

(i) $\vec{r}(t) = \langle \cos(t), -\sin(t), t \rangle$

$\vec{r}\,'(t) = \langle -\sin(t), -\cos(t), 1 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(-\sin(t))^2 + (-\cos(t))^2 + (1)^2}$

$= \sqrt{\sin^2(t) + \cos^2(t) + 1}$

$= \sqrt{1+1} = \sqrt{2}$

$T(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \langle \frac{-\sin(t)}{\sqrt{2}}, \frac{-\cos(t)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

(ii) $T(\pi) = \langle -\frac{1}{\sqrt{2}}\sin(\pi), -\frac{1}{\sqrt{2}}\cos(\pi), \frac{1}{\sqrt{2}} \rangle$

$= \langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

sometimes it is easier to find $\vec{r}\,'(a)$ and $\|\vec{r}\,'(a)\|$ separately and combine them for $T(a)$

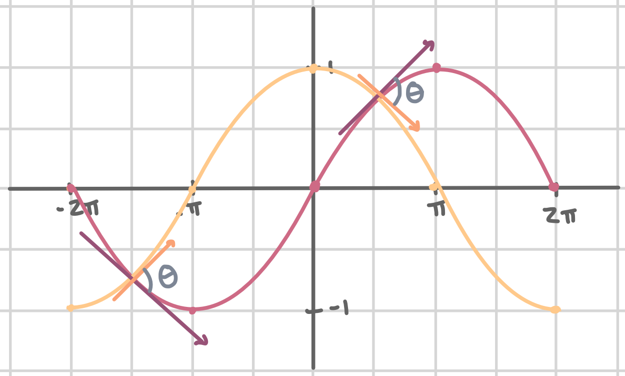
$P = \vec{r}(\pi) = \langle \cos(\pi), -\sin(\pi), \pi \rangle$

$= \langle 1, 0, \pi \rangle$

tangent line: $T_\pi(t) = \langle 1, 0, \pi \rangle + t \langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

angle of intersection

Suppose two curves $\vec{r}_1$ and $\vec{r}_2$ intersect at a point P. Then the angle they intersect at can be determined by finding the angle of intersection of the tangent vector vectors at the point P.



Curves can intersect multiple times (sometimes with different angles) so we must be careful when inputting the intersection point. Curves can also run at different "speeds" i.e. the parameters could be different for each curve at the point of intersection.

example. Find the angle of intersection for the curves $\vec{r}_1(t) = \langle \cos(t), -\sin(t), t \rangle$ and $\vec{r}_2(s) = \langle -s, s^2-1, \ln(s)+\pi \rangle$ at the point $P = (-1, 0, \pi)$

First solve for the values t and v: $\vec{r}_1(\pi) = \langle \cos(\pi), -\sin(\pi), \pi \rangle = \langle -1, 0, \pi \rangle$ & $\vec{r}_2(1) = \langle -1, (1)^2-1, \ln(1)+\pi \rangle = \langle -1, 0, \pi \rangle$.

Next find the tangent vectors: $\vec{r}_1'(t) = \langle -\sin(t), -\cos(t), 1 \rangle$ & $\vec{r}_2'(s) = \langle -1, 2s, \frac{1}{s} \rangle$

Tangent vectors at our given t & s: $\vec{r}_1'(\pi) = \langle 0, 1, 1 \rangle$ & $\vec{r}_2'(1) = \langle -1, 2, 1 \rangle$

Angle between tangents at P: $\vec{r}_1 \cdot \vec{r}_2 = \|\vec{r}_1\| \cdot \|\vec{r}_2\| \cdot \cos(\theta)$

$$\langle 0, -1, 1 \rangle \cdot \langle -1, 2, 1 \rangle = \|\langle 0, -1, 1 \rangle\| \cdot \|\langle -1, 2, 1 \rangle\| \cdot \cos(\theta)$$

$$(0)(-1) + (-1)(2) + (1)(1) = \sqrt{(0)^2 + (-1)^2 + (1)^2}\sqrt{(-1)^2 + (2)^2 + (1)^2}\cos(\theta)$$

$$0 - 2 + 1 = \sqrt{2}\sqrt{6}\cos\theta$$

$$\frac{-1}{2\sqrt{3}} = \cos(\theta)$$

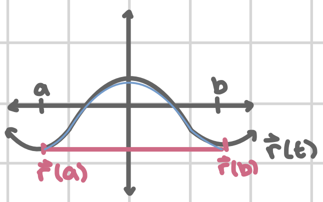
$$\theta = \arccos\left(-\frac{\sqrt{3}}{6}\right)$$

arclength

There are two types of distance that are commonly discussed:

- displacement - the "direct" or shortest distance between the starting point and end point.
- total distance traveled - distance that takes into account the path followed.

Here is a 2D photo to show why both are important.



$|\vec{r}(b) - \vec{r}(b)|$ = the two points $\vec{r}(b)$ & $\vec{r}(a)$ subtracted

$\int_a^b \|\vec{r}\,'(t)\|\,dt$ = the distance traveled along the curve $\vec{r}(t)$

In Calculus II, we found that the arclength for a two-dimensional curve is given by $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$. The natural extension to three-dimensions is $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$. We can simplify this equation to be $L = \int_a^b \|\vec{r}\,'(t)\|\,dt$.

example. Find the arclength of $\vec{r}(t) = \langle t, 3\cos t, 3\sin t \rangle$ where $-5 \le t \le 5$.

$\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t \rangle$

$|\vec{r}\,'(t)| = \sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2} = \sqrt{1+9} = \sqrt{10}$

$L = \int_{-5}^{5} \sqrt{10}\,dt = \sqrt{10}(5-(-5)) = 10\sqrt{10}$

The last concept is the arclength function which tells us the distance traveled at time t. We define the arclength function as $s(t) = \int_0^t \|\vec{r}\,'(t)\|\,dt$.

example. Determine the arclength function for $\vec{r}(t) = \langle t, 3\cos(t), 3\sin(t) \rangle$.

$\vec{r}\,'(t) = \langle 1, -3\sin(t), 3\cos(t) \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{10}$

$s(t) = \int_0^t \sqrt{10}\,dt = t\sqrt{10}\,\Big|_0^t = t\sqrt{10}$

We can also ask, where are we on the curve if we have traveled a specified distance? To find this we solve the arclength function for t and plug the result into the parameterization.

example. Reparameterize the function into $\vec{r}(t(s))$.

$s = t\sqrt{10} \implies t = \frac{s}{\sqrt{10}}$

$\vec{r}(s(t)) = \langle \frac{s}{\sqrt{10}}, 3\cos(\frac{3}{\sqrt{10}}), 3\cos(\frac{3}{\sqrt{10}}) \rangle$

Standard 06: TNB Frame, Normal Plane, and Osculating Plane

Tangent, Normal, and Binormal Vectors

In this section we want to look at an application of derivatives for vector-valued functions. We build on an application we saw last time: the unit tangent vector.

unit tangent vector

Provided $\vec{r}'(t) \neq \vec{0}$, the unit tangent vector to the curve is given by $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$.

example. Find the general formula for the unit tangent vector to the curve given by $\vec{r}(t) = \langle t, 3\sin(t), 3\cos(t) \rangle$

$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$

$\vec{r}'(t) = \langle 1, 3\cos(t), -3\sin(t) \rangle$

$$\begin{aligned}\|\vec{r}'(t)\| &= \sqrt{(1)^2 + (3\cos(t))^2 + (-3\sin(t))^2} \\ &= \sqrt{1 + 9\cos^2(t) + 9\sin^2(t)} \\ &= \sqrt{1 + 9(\cos^2(t) + \sin^2(t))} \\ &= \sqrt{1+9} = \sqrt{10}\end{aligned}$$

$\vec{T}'(t) = \langle \frac{1}{\sqrt{10}}, \frac{3\cos(t)}{\sqrt{10}}, \frac{-3\sin(t)}{\sqrt{10}} \rangle$

unit normal vector

Similarly, the unit normal vector to the curve is defined to be $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$.

example. Find the general formula for the unit normal vector to the curve given by $\vec{r}'(t) = \langle t, 3\sin(t), 3\cos(t) \rangle$.

$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$, $\vec{T}(t) = \langle \frac{1}{\sqrt{10}}, \frac{3\cos(t)}{\sqrt{10}}, \frac{-3\sin(t)}{\sqrt{10}} \rangle$

$\vec{T}'(t) = \langle 0, \frac{-3}{\sqrt{10}}\sin(t), \frac{-3}{\sqrt{10}}\cos(t) \rangle$

$$\begin{aligned}\|\vec{T}'(t)\| &= \sqrt{0^2 + (\tfrac{-3}{\sqrt{10}}\sin(t))^2 + (\tfrac{-3}{\sqrt{10}}\cos(t))^2} \\ &= \sqrt{\tfrac{9}{10}\sin^2(t) + \tfrac{9}{10}\cos^2(t)} \\ &= \sqrt{\tfrac{9}{10}(\sin^2(t) + \cos^2(t))} \\ &= \sqrt{\tfrac{9}{10}} = \tfrac{3}{\sqrt{10}}\end{aligned}$$

$$\begin{aligned}\vec{N}(t) &= \langle \frac{0}{3/\sqrt{10}}, \frac{-3\sin(t)/\sqrt{10}}{3/\sqrt{10}}, \frac{-3\cos(t)/\sqrt{10}}{3/\sqrt{10}} \rangle \\ &= \langle 0, -\sin(t), -\cos(t) \rangle\end{aligned}$$

Fun Facts about $\vec{N}(t)$

- $\vec{N}(t) \perp \vec{T}(t)$
- $\vec{N}(t) \perp \vec{r}(t)$
- $\vec{N}(t)$ should remind you of $\vec{n} = \langle a, b, c \rangle$
- If $\|\vec{r}(t)\| = c$ for all t then $\vec{r}'(t) \perp \vec{r}(t)$

ex. $\|\vec{T}(t)\| = 1$ for all t then $\vec{N}(t) \perp \vec{T}(t)$

unit binormal vector

Lastly, we define the unit binormal vector of a curve to be $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.

Big Terrible Nightmare

example. Find the general formula for the unit binormal for the curve given by $\vec{r}(t) = \langle t, 3\sin(t), 3\cos(t) \rangle$.

$\vec{B} = \vec{T}(t) \times \vec{N}(t)$, $\vec{T}(t) = \langle \frac{1}{\sqrt{10}}, \frac{3\cos(t)}{\sqrt{10}}, \frac{-3\sin(t)}{\sqrt{10}} \rangle$, $\vec{N}(t) = \langle 0, -\sin(t), -\cos(t) \rangle$

$$\begin{aligned}\vec{B}(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{10}} & \frac{3\cos(t)}{\sqrt{10}} & \frac{-3\sin(t)}{\sqrt{10}} \\ 0 & -\sin(t) & -\cos(t) \end{vmatrix} \\ &= \left(\tfrac{-3}{\sqrt{10}}\cos^2(t) - \tfrac{3}{\sqrt{10}}\sin^2(t)\right)\vec{i} - \left(\tfrac{-1}{\sqrt{10}}\cos(t) - 0\right)\vec{j} + \left(\tfrac{-1}{\sqrt{10}}\sin(t) - 0\right)\vec{k} \\ &= \left(\tfrac{-3}{\sqrt{10}}(\cos^2(t) + \sin^2(t))\right)\vec{i} + \left(\tfrac{1}{\sqrt{10}}\cos(t)\right)\vec{j} - \left(\tfrac{1}{\sqrt{10}}\sin(t)\right)\vec{k} \\ &= \tfrac{-3}{\sqrt{10}}\vec{i} + \tfrac{1}{\sqrt{10}}\cos(t)\vec{j} + \tfrac{1}{\sqrt{10}}\sin(t)\vec{k} \\ &= \langle \tfrac{-3}{\sqrt{10}}, \tfrac{1}{\sqrt{10}}\cos(t), \tfrac{1}{\sqrt{10}}\sin(t) \rangle\end{aligned}$$

Alternative Formulas:

- $\vec{B}(t) = \frac{\vec{r}'(t) \times \vec{r}''(t)}{\|\vec{r}'(t) \times \vec{r}''(t)\|}$
- $\vec{N}(t) = \vec{B}(t) \times \vec{T}(t)$

Not Bad Today