Standard07: Partial Derivatives

partial derivatives

Recall that given a function of one variable, $f(x)$, the derivative, $f'(x)$, represents the rate of change of the function as x changes. The issue, we have more than one variable to vary. If we allow more than one to vary then we have an infinite number of ways we can change them: same speed, one faster than the other, different degrees of faster, etc. In this section we concentrate on changing only one variable at a time as the remaining variable(s) are held fixed.

In practice, the partial derivative of $f=f(x,y)$ with respect to x is the derivative of f with respect to x while treating all other variable(s) as a constant. We denote the partial with respect to x as f_x. We can also define a partial with respect to y similarly: take the derivative of f with respect to y while treating all other variables as constants. This definition can be extended to a function with more than two variables. You can also take higher partial derivatives; $(f_x)_x$, $(f_x)_y$, $(f_y)_y$, $(f_y)_x$. Alternative notation: $f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f(x,y)) = z_x = \frac{\partial z}{\partial x} = D_x f$.

example. Compute the second partial derivatives of the following functions:

(i) $f(x,y) = x^2 + y^2 + xy$

$f_x = 2x + 0 + y$
$f_y = 0 + 2y + x$
$(f_x)_x = 2 + 0$
$(f_x)_y = 0 + 1$
$(f_y)_x = 0 + 1$
$(f_y)_y = 2 + 0$

(ii) $h(s,t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}$

$f_t = 7t^6 \ln(s^2) - \frac{27}{t^4} + 0$
$f_s = t^7 \frac{2}{s} + 0 - \frac{4}{7} s^{-3/7}$
$(f_t)_t = 42t^5 \ln(s^2) + \frac{108}{t^5}$
$(f_t)_s = 7t^6 \frac{2}{s} + 0$
$(f_s)_t = 7t^6 \cdot \frac{2}{s} + 0$
$(f_s)_s = -t^7 \cdot \frac{2}{s^{-2}} + \frac{12}{14} s^{-10/7}$

(iii) $g(x,y,z) = \frac{x\sin(y)}{z^2}$

$g_x = \frac{\sin(y)}{z^2}$
$g_y = \frac{x}{z^2}\cos(y)$
$g_z = -2\frac{x\sin(y)}{z^3}$
$(g_x)_x = 0$
$(g_x)_y = \frac{1}{z^2}\cos(y)$
$(g_x)_z = -2\frac{\sin(y)}{z^3}$
$(g_y)_x = \frac{\cos(y)}{z^2}$
$(g_y)_y = -\frac{x}{z^2}\cos(y)$
$(g_y)_z = -2\frac{x}{z^{-3}}\cos(y)$
$(g_z)_x = \frac{-2\sin(y)}{z^2}$
$(g_z)_y = \frac{-2x\cos(y)}{z^3}$
$(g_z)_z = 6\frac{x\sin(y)}{z^3}$

Clairaut's Theorem. Suppose that f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yz} are continuous on this disk then $f_{xy}(a,b) = f_{yx}(a,b)$.

gradient

There is a special vector called the gradient vector of $f=f(x,y)$ defined by $\nabla f = \langle f_x, f_y \rangle$. This can be extended.

example. Find the gradient of the following vectors:

(i) $f(x,y) = x^2 + y^2 + xy$

$\nabla f = \langle 2x + y, 2y + x \rangle$

(ii) $h(s,t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}$

$\nabla f = \langle 7t^6 \ln(s^2) - 27\frac{1}{t^4}, t^7 \cdot \frac{2}{s} + \frac{4}{7} s^{-3/7} \rangle$

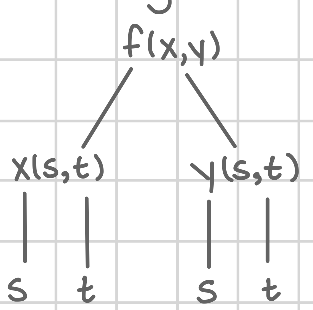
(iii) $g(x,y,z) = \frac{x\sin(y)}{z^2}$

$\nabla f = \langle \frac{\sin(y)}{z^2}, \frac{x}{z^2}\cos(y), \frac{-2x\sin(y)}{z^2}$

chain rule

The name chain rule should immediately bring to mind the chain rule for derivative of single variable functions: if $F(x) = f(g(x))$ then $F'(x) = f'(g(x)) \cdot g'(x)$. Alternative notation for this same phenomenon is: if $y = f(x)$ and $x = g(t)$ then $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$. We want to extend this concept to multi-variable.

Given a function of two variables $f(x,y)$ and two input functions $x(s,t)$ and $y(s,t)$, we can set up the following tree:



We can now use this diagram to compute the partial derivatives $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$:
To find $\frac{\partial f}{\partial s}$ we move from the f at the top of the tree down to the two s's.
Each time we go down a line we take the derivative of the top over the bottom.
If we continue down the tree we multiple the partial derivatives.
If we need to reset to the f to head down a new path then we put a plus between the terms.

Following these rules we have $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$ and $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$. This can be extended to $f(x,y,z)$.

example. Suppose that $z=f(x,y)$ is a function of x and y and we have input functions $x=s^2-t^2$ and $y=2st$. If $\frac{\partial z}{\partial s}=10$ and $\frac{\partial z}{\partial t}=0$ when $(s,t)=(1,2)$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$10 = \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{\partial z}{\partial x}\cdot(2s) + \frac{\partial z}{\partial y}\cdot(2t)$$

at $(1,2)$ $= \frac{\partial z}{\partial x}\cdot(2) + \frac{\partial z}{\partial y}(4)$

$$0 = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\partial z}{\partial x}\cdot(-2t) + \frac{\partial z}{\partial y}\cdot(2s)$$

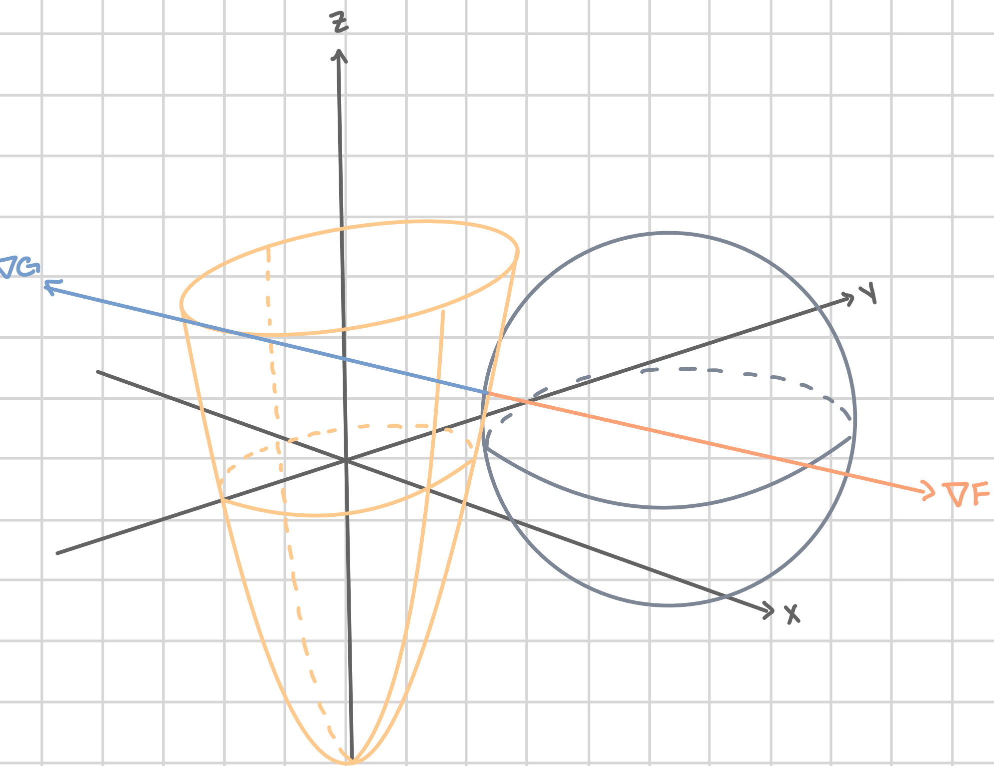
at $(1,2)$ $= \frac{\partial z}{\partial x}\cdot(-4) + \frac{\partial z}{\partial y}=(2)$

Solve the linear equations $10 = 2\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y}$ and $0 = -4\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y}$ $\Rightarrow$ $\frac{\partial z}{\partial x}=1$ and $\frac{\partial z}{\partial y}=2$.

application

The gradient gives the normal vector of the tangent plane at a specific point. We can use the information in this standard to solve the following problem:

example. Show that the paraboloid $2x^2+y^2-z=5$ and the sphere $(x-3)^2+(y-4)^2+(z-½)^2=\frac{33}{4}$ are tangent to each other at the point $(1,2,1)$. Find a plane tangent to both surfaces at $(1,2,1)$.



$F(x,y,z) = 2x^2+y^2-z$

$\nabla F = \langle 4x, 2y, -1\rangle$

$\nabla F(1,2,1) = \langle 4, 4, -1\rangle$

$G(x,y,z) = x^2-6x+y^2-8y+z^2-z+\frac{101}{4}$

$\nabla G = \langle 2x-6, 2y-8, 2z-1\rangle$

$\nabla G(1,2,1) = \langle -4,-4,1\rangle$

Since $\langle 4,4,-1\rangle = -1\langle -4,-4,1\rangle$ so ∇F and ∇G are parallel. Therefore they are tangent at $(1,2,1)$.

tangent plane: $4(x-1)+4(y-2)-(z-1)=0$

$4x+4y-z=1$