Standard 08: Directional Derivatives

directional derivatives

Up until now we have seen two partial derivatives, f_x and f_y. These derivatives represent the rate of change of f as we vary x while holding y fixed and the rate of change as we vary y while holding x fixed, respectively. We now want to find the rate of change of f as we vary x and y simultaneously. As mentioned previously, there are many ways to vary both x and y: different speeds, different directions, etc.

Suppose we want the rate of change of f at the point (x_0, y_0) and we want both x and y to increase but x to increase twice as fast as y. We can model this change using position vectors, start at $(0,0)$ and increase x twice as much as y. But wait! $\langle 2,1\rangle$, $\langle \frac{1}{5}, \frac{1}{10}\rangle$, $\langle 6,3\rangle$ all meet the criteria. We need a way to consistently find the rate of change in a direction.

We do this by insisting that the vector that defines the direction of change be a unit vector. For our example the unit vector is $\vec{u} = \langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$. In some cases the direction of change may be given as an angle, when this happens we use the unit vector $\vec{u} = \langle \cos(\theta), \sin(\theta)\rangle$.

Now that we have discussed the direction of change in depth, we can now find the rate of change of f in a specified direction:
The rate of change of $f(x,y)$ in the direction of the unit vector $\vec{u} = \langle a,b\rangle$ is called the directional derivative and is found by $D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b = \nabla f(x,y) \cdot \vec{u}$.

example. Find the directional derivative of the following functions in the matching direction:

(a) $f(x,y) = ye^x + y^2$, $\theta = \frac{\pi}{2}$

$\vec{u} = \langle \cos(\frac{\pi}{2}), \sin(\frac{\pi}{2})\rangle = \langle 0,1\rangle$

$\nabla f(x,y) = \langle ye^x, e^x + 2y\rangle$

$D_{\vec{u}} f = 0 \cdot ye^x + 1 \cdot e^x + 2y$

$= e^x + 2y$

(b) $f(x,y,z) = x^2 z + y^3 z - xyz$, $\vec{v} = \langle -1,0,1\rangle$

$\vec{u} = \frac{1}{\sqrt{(-1)^2+(0)^2+(1)^2}} \langle -1,0,1\rangle = \langle \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\rangle$

$\nabla f = \langle 2xz - yz, 3y^2 z - xz, x^2 + y^3 - xy\rangle$

$D_{\vec{u}} f = -1 \cdot (2xz - yz) + 0 \cdot (3y^2 z - xz) + 1 \cdot (x^2 + y^3 - xy)$

$= -2xz + yz + x^2 + y^3 - xy$

maximization

The maximum value of $D_{\vec{u}} f$ is given by $\|\nabla f\|$ and will occur in the direction given by ∇f.

proof. Using the dot product version of the directional derivative, the formula of the dot product with $\cos\theta$, and the fact that the magnitude of a unit vector is 1, we have $D_{\vec{u}} f = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos(\theta) = \|\nabla f\| \cdot \cos(\theta)$. To maximize the formula we must maximize $\cos(\theta)$ which has a range of $[-1,1]$. The maximum value of $\cos(\theta)$ occurs at $\theta = 0$. Therefore the maximum value of $D_{\vec{u}} f$ is $\|\nabla f\|$ and occurs when the angle between ∇f and $\vec{u}$ is 0 i.e. in the direction of the gradient. Similarly, the minimum value occurs when $\cos(\theta) = -1$ i.e. has value $-\|\nabla f\|$ in the direction $-\nabla f$.

example. Find the maximal rate of change of the following functions at the point given:

(a) $f(x,y) = ye^x + y^2$, $(a,b) = (0,1)$

$\nabla f = \langle ye^x, e^x + 2y\rangle$

$\nabla f(0,1) = \langle 1, 1 + 2(1)\rangle = \langle 1,3\rangle$

$\|\nabla f(0,1)\| = \sqrt{(1)^2 + (3)^2} = \sqrt{10}$

(b) $f(x,y,z) = x^2 z + y^3 z - xyz$, $(a,b,c) = (1,0,1)$

$\nabla f = \langle 2xz - yz, 3yz - xz, x^2 + y^3 - xy\rangle$

$\nabla f(1,0,1) = \langle 2(1)(1) - (0)(1), 3(0)(1) - (1)(1), (1)^2 + (0)^2 - (0)(1)\rangle = \langle 2,-1,0\rangle$

$\|\nabla f(1,0,1)\| = \sqrt{(2)^2 + (-1)^2 + (0)^2} = \sqrt{5}$