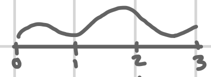
Standard 10: Lagrange Multipliers

Finding extremas on a bounded area can be long and very tedious, as seen in the example in the last standard. It is rather simple to find critical points and test them. It is much harder to test if there are any extremas on the boundary that do not register as a critical point; for example  has critical points at (roughly) $1/2$, 1, $7/4$, & $5/2$ but if the boundary constraint was $[0, 3/2]$ then our absolute max would be $3/2$ and not a critical point. In order to shorten the process, we will use an optimization technique.

Lagrange multipliers

We want to optimize a function $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$. The method goes like this:

1. Solve the following system of equations: $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$
   $g(x,y,z) = k$
2. Plug in all solutions (x,y,z) from the first step into $f(x,y,z)$ and identify the extrema values, provided they exist and $\nabla g \neq \vec{0}$.

This works because the extrema values of $f(x,y,z)$ will appear when the surface $f(x,y,z)=k$ and the surface of the constraint are parallel i.e. their gradients are multiples of each other.

example. Find the maximum and minimum of $f(x,y) = 5x - 3y$ subject to the constraint $x^2+y^2=136$.

Since $x^2+y^2=136$ is a closed and bounded region, the extreme value theorem tells us that a min/max must exists. We can start the Lagrange multiplier method:

1. Solve $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y)=k$

$\nabla f(x,y) = \langle 5, -3 \rangle$

$\nabla g(x,y) = \langle 2x, 2y \rangle$

$\Rightarrow$ $5 = \lambda 2x$

$-3 = \lambda 2y$

$\Rightarrow$ $x = \frac{5}{2\lambda}$

$y = \frac{-3}{2\lambda}$

Plug into constraint:

$(\frac{5}{2\lambda})^2 + (\frac{-3}{2\lambda})^2 = 136$

$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = 136$

$\frac{34}{4\lambda^2} = 136$

$\frac{1}{\lambda^2} = 16$

$\lambda^2 = \frac{1}{16}$

$\lambda = \pm\frac{1}{4}$

Find points:

- $\lambda = +\frac{1}{4}$
  $x = \frac{5}{2(1/4)} = 10$ $\quad$ $y = \frac{-3}{2(1/4)} = -6$
- $\lambda = -\frac{1}{4}$
  $x = \frac{5}{2(-1/4)} = -10$ $\quad$ $y = \frac{-3}{2(-1/4)} = 6$

2. Plug into $f(x,y)$

$f(10,-6) = 5(10) - 3(-6) = 68$ $\quad$ maximum at $(10,-6)$ of 68

$f(-10,6) = 5(-10) - 3(6) = -68$ $\quad$ minimum at $(-10,6)$ of -68

We can also have inequalities as constraints. The process is nearly identical except we must also consider all critical points that satisfy the inequalities.

example. Find the maximum and minimum values of $f(x,y) = 4x^2 + 10y^2$ on the disk $x^2+y^2 \leq 4$.

The extreme value theorem tells us that the closed and bounded must contain a max/min.

First, find critical points that satisfy the inequality

$f_x = 8x \Rightarrow 8x = 0 \Rightarrow x = 0$

$f_y = 20y \quad 20y = 0 \quad y = 0$

Second, use the Lagrange multipliers

1. Solve $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = k$

$8x = 2\lambda x$ $\quad$ $8x - 2\lambda x = 0$ $\quad$ if $x = 0$ $\quad$ if $\lambda = 4$ $\quad$ then

$20y = 2\lambda y \Rightarrow 2x(4-\lambda) = 0 \Rightarrow (0)^2 + y^2 = 4$ $\quad$ $20y = 2(4)y$ $\quad$ $x^2 + (0)^2 = 4$

$x^2 + y^2 = 4$ $\quad$ $x = 0, \lambda = 4$ $\quad$ $y = \pm 2$ $\quad$ $12y = 0$ $\quad$ $x = \pm 2$

$\Rightarrow$ $(0,2), (0,-2), (2,0), (-2,0)$

2. Plug into $f(x,y)$

$f(0,0) = 4(0)^2 + 10(0)^2 = 0$

$f(0,2) = 4(0)^2 + 10(2)^2 = 40$

$f(0,-2) = 4(0)^2 + 10(-2)^2 = 40$

$f(2,0) = 4(2)^2 + 10(0)^2 = 16$

$f(-2,0) = 4(-2)^2 + 10(0)^2 = 16$

minimum of 0 at $(0,0)$

maximum of 40 at $(0,2)$ & $(0,-2)$

If you have two constraints then the system of equations becomes $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$, $g(x,y,z) = k$, $h(x,y,z) = c$.

example. We want to optimize $f(x,y,z) = yz + xz$ subject to two constraints: $xz = 1$ and $y^2 + z^2 = 1$.

1. Solve the two constraint system

$$\begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g(x,y,z) = k \\ h(x,y,z) = c \end{cases} \qquad \begin{cases} z = \lambda \cdot z + \mu \cdot 0 \\ z = \lambda \cdot 0 + \mu\, 2y \\ x = \lambda \cdot z + \mu \cdot 2z \\ xz = 1 \\ y^2 + z^2 = 1 \end{cases} \qquad \begin{cases} z - \lambda z = 0 \\ z = 2\mu y \\ y + x = \lambda x + 2\mu z \\ xz = 1 \\ y^2 + z^2 = 1 \end{cases} \quad z(1-\lambda) = 0$$

Equation 1 gives $z = 0$ or $\lambda = 1$, but equation 4 gives $z \neq 0$:

$$\begin{cases} \lambda = 1 \\ z = 2\mu y \\ y + x = (1)x + 2\mu z \\ xz = 1 \\ y^2 + z^2 = 1 \end{cases}$$

Is it closed and bounded?
- $xz = 1$ is not bounded by itself
- when $xz = 1$, $f(x,y,z) = yz + xz = yz + 1$
- $y^2 + z^2$ bounds y and z

$\Rightarrow f(x,y,z) = yz + 1$ is bounded

Equation 3 gives $y = 2\mu z$:

$$\begin{cases} \lambda = 1 \\ z = 2\mu(2\mu z) \quad z - 4\mu^2 z = 0 \quad z(1 - 4\mu^2) = 0 \\ y = 2\mu z \\ xz = 1 \\ y^2 + z^2 = 1 \end{cases}$$

Equation 5 gives $z = \pm \frac{1}{\sqrt{2}}$:

case 1: $\mu = 1/2$

$$\begin{cases} \lambda = 1 \\ \mu = 1/2 \\ y = 2(1/2)z \quad y = z \\ xz = 1 \\ y^2 + z^2 = 1 \end{cases}$$

case 2: $\mu = -1/2$

$$\begin{cases} \lambda = 1 \\ \mu = -1/2 \\ y = 2(-1/2)z \quad y = -z \\ xz = 1 \\ y^2 + z^2 = 1 \end{cases}$$

Equation 3 put into equation 5 gives:

case 1a: $z = 1/\sqrt{2}$

$$\begin{cases} \lambda = 1 \\ \mu = 1/2 \\ y = z \\ xz = 1 \\ z = 1/\sqrt{2} \end{cases}$$

case 1b: $z = -1/\sqrt{2}$

$$\begin{cases} \lambda = 1 \\ \mu = 1/2 \\ y = z \\ xz = 1 \\ z = -1/\sqrt{2} \end{cases}$$

case 2a: $z = 1/\sqrt{2}$

$$\begin{cases} \lambda = 1 \\ \mu = -1/2 \\ y = -z \\ xz = 1 \\ z = 1/\sqrt{2} \end{cases}$$

case 2b: $z = -1/\sqrt{2}$

$$\begin{cases} \lambda = 1 \\ \mu = -1/2 \\ y = -z \\ xz = 1 \\ z = -1/\sqrt{2} \end{cases}$$

2. Plug into $f(x,y,z)$

$f(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 3/2$

$f(-\sqrt{2}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = 3/2$

$f(\sqrt{2}, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 1/2$

$f(-\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) = 1/2$

maximum value $= 3/2$, minimum value $= 1/2$