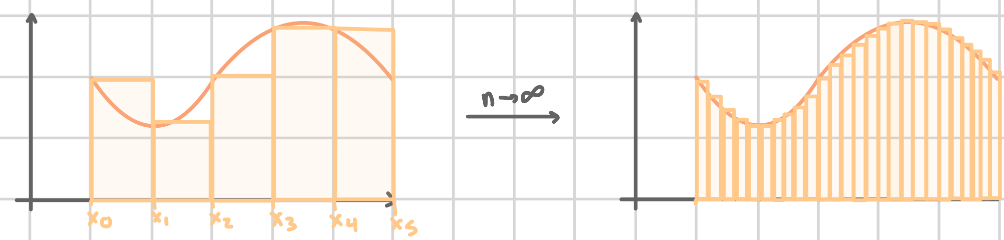
Standard 11: Double Integrals

Double Integration

Now that we have completed the applications of partial derivatives, we must find a way to reverse them. In calculus II we use integration to reverse taking the derivative. Today we define double integration to do a similar thing for partial derivatives. Similar to the partial derivative, we treat one variable as a constant as we integrate with respect to the other.

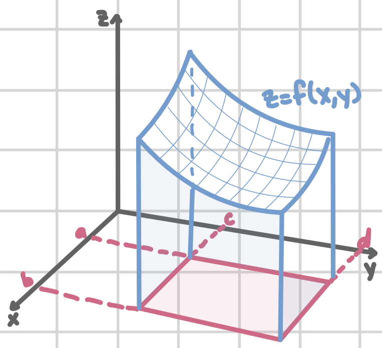
single variable review

Recall that $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ gave the area under the curve $f(x)$ for $a \le x \le b$ by summing up an "infinite" number of rectangles under the curve.



rectangular regions

In this section we aim to integrate a function of two variables, $f(x,y)$. In single-variable we integrated over an interval, for two-variables it makes sense to move up to a two-dimension region. We start with a rectangle.



For the region $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$, we define the double integral to be $\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$. Similar to the single-variable integral finding the area under the curve, the double integral will give the volume under the surface by summing up an infinite number of tiny rectangles under the surface. This means $\iint_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i,y_i)\,\Delta A$.

Note that the inner differential matches up with the limits on the inner integral and the same follows for the outer, i.e. if the inner differential is dy then the limits on the inner integral must have y limits.

To compute the double integral we use the same thought process as partial derivatives and work our way out: $\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ so we can first compute $\int_c^d f(x,y)\,dy$ by holding x constant and integrating with respect to y.

example. Compute $\iint_R x^2y^2 + \cos(\pi x) + \sin(\pi y)$ over $R = [2,-1] \times [0,1]$.

$\iint_R x^2y^2 + \cos(\pi x) + \sin(\pi y)\,dA = \int_0^1 \int_{-2}^{-1} x^2y^2 + \cos(\pi x) + \sin(\pi y)\,dx\,dy = \int_{-2}^{-1}\int_0^1 x^2y^2 + \cos(\pi x) + \sin(\pi y)\,dy\,dx$

$\sin(\pi y)$ is a constant w.r.t x

$= \int_0^1 \left[\frac{1}{3}x^3y^2 + \frac{1}{\pi}\sin(\pi x) + \sin(\pi y)\cdot x\right]_{-2}^{-1} dy = \int_{-2}^{-1}\left[\frac{1}{3}x^2y^3 + y\cos(\pi x) - \frac{1}{\pi}\cos(\pi y)\right]_0^1 dy$

$= \int_0^2 \frac{1}{3}(-1)^3y^2 + \frac{1}{\pi}\sin(-\pi) + (-1)\sin(\pi y) - \left(\frac{1}{3}(-2)^3y^2 + \sin(-2\pi) - 2\sin(\pi y)\right) dy = \int_{-2}^{-1} \frac{1}{3}x^2 + \cos(\pi x) + \frac{1}{\pi} - (0 + 0 - \frac{1}{\pi})\,dy$

$= \int_0^1 \frac{7}{3}y^2 + \sin(\pi y)\,dy = \int_{-2}^{-1} \frac{1}{3}x^2 + \cos(\pi x) + \frac{2}{\pi}\,dy$

$= \left[\frac{7}{9}y^3 - \frac{1}{\pi}\cos(\pi y)\right]_0^1 = \left[\frac{1}{9}x^3 + \frac{1}{\pi}\sin(\pi x) + \frac{2}{\pi}y\right]_{-2}^{-1}$

$= \frac{7}{9}(1)^3 - \frac{1}{\pi}\cos(\pi) - \left(\frac{7}{9}(0)^3 - \frac{1}{\pi}\cos(0)\right) = \frac{1}{9}(-1)^3 + \frac{1}{\pi}\sin(-\pi) + \frac{2}{\pi}(-1) - \left(\frac{1}{9}(-2)^3 + \frac{1}{\pi}\sin(-2\pi) + \frac{2}{\pi}(-2)\right)$

$= \frac{7}{9} + \frac{2}{\pi} = \frac{7}{9} + \frac{2}{\pi}$

Important facts:

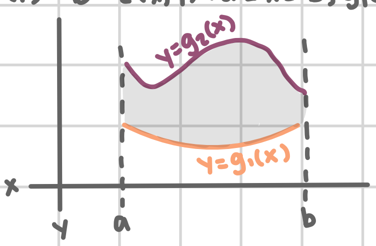
- In the case of an indefinite integral, we replace the $+c$ with $+$ a function of the other variable as it is a constant.
  - ↳ For example $f(x,y) = x^3 + 2y \Rightarrow f_x = 3x^2 + 0$ so $\int 3x^2\,dx$ needs to be $x^3 + g(y)$
- When dealing with a rectangular region, $\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$
  - ↳ for non-rectangular regions there will be more work
- If $f(x,y) = g(x)\cdot h(y)$ and we are integrating over the rectangle $R = [a,b] \times [c,d]$ then,

  $\iint_R f(x,y)\,dA = \iint_R g(x)\,h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$

general regions

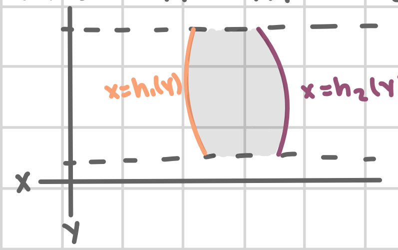
Most regions are not a rectangle so we need to consider $\iint_R f(x,y)\,dA$ where R is any region. Any region R can be described in one of two ways:

(i) $D=\{(x,y) \mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$



$$\iint_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(ii) $D=\{(x,y) \mid h_1(y)\le x\le h_2(y),\ c\le y\le d\}$

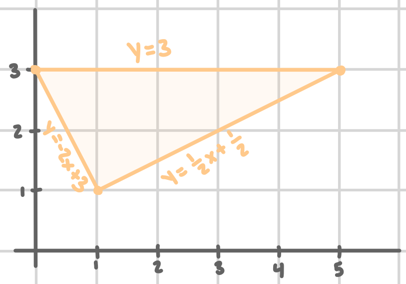


$$\iint_D f(x,y)\,dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Properties:

1. $\iint_R f(x,y)+g(x,y)\,dA=\iint_R f(x,y)\,dA+\iint_R$
2. $\iint_R c\cdot f(x,y)\,dA=c\cdot\iint_R f(x,y)\,dA$ where c is any constant.
3. If the region R can be split into two separate regions R_1 and R_2 then $\iint_R f(x,y)\,dA=\iint_{R_1} f(x,y)\,dA+\iint_R f(x,y)\,dA$

example. Compute $\iint_R xe^y\,dA$ where R is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$.



The region can be described two ways:

vertically: $R=R_1\cup R_2$

$R_1=\{(x,y) \mid 0\le x\le 1,\ -2x+3\le y\le 3\}$

$R_2=\{(x,y) \mid 1\le x\le 5,\ \frac{1}{2}x+\frac{1}{2}\le y\le 3\}$

horizontally: $R=\{(x,y) \mid -\frac{1}{2}y+\frac{3}{2}\le x\le 2y-1,\ 1\le y\le 3\}$

$y=-2x+3 \implies x=-\frac{1}{2}y+\frac{3}{2}$

$y=\frac{1}{2}x+\frac{1}{2} \implies x=2y-1$

$$\iint_R xe^y\,dA=\iint_{R_1} xe^y\,dA+\iint_{R_2} xe^y\,dA$$
$$=\int_0^1\int_{-2x+3}^3 xe^y\,dy\,dx+\int_1^5\int_{\frac{1}{2}x+\frac{1}{2}}^3 xe^y\,dy\,dx$$
$$=\int_0^1[xe^y]_{-2x+3}^3\,dx+\int_1^5[xe^y]_{\frac{1}{2}x+\frac{1}{2}}^3\,dx$$
$$=\int_0^1 xe^3-\overset{①}{xe^{-2x+3}}\,dx+\int_1^5 \overset{②}{xe^{\frac{1}{2}x+\frac{1}{2}}}\,dx$$

① $\int xe^{-2x+3}\,dx$

$u=x$ $\quad dv=e^{-2x+3}$

$du=1dx$ $\quad v=\frac{1}{-2}e^{-2x+3}$

$(x)(\frac{1}{-2}e^{-2x+3})-\int\frac{1}{-2}e^{-2x+3}\,dx$

$-\frac{1}{2}xe^{-2x+3}-(\frac{1}{4}e^{-2x+3})$

② $\int_1^5 xe^{\frac{1}{2}x+\frac{1}{2}}\,dx$

$u=x$ $\quad dv=e^{\frac{1}{2}x+\frac{1}{2}}$

$du=1dx$ $\quad v=2e^{\frac{1}{2}x+\frac{1}{2}}$

$(x)(2e^{\frac{1}{2}x+\frac{1}{2}})-\int 2e^{\frac{1}{2}x+\frac{1}{2}}\,dx$

$2xe^{\frac{1}{2}x+\frac{1}{2}}-(4e^{\frac{1}{2}x+\frac{1}{2}})$

$$=\frac{e^3}{2}x^2-(-\tfrac{1}{2}xe^{-2x+3}-\tfrac{1}{4}e^{-2x+3})\Big]_0^1+2xe^{\frac{1}{2}x+\frac{1}{2}}-4e^{\frac{1}{2}x+\frac{1}{2}}\Big]_1^5$$
$$=\frac{e^3}{2}+\frac{1}{2}e^1+\frac{1}{4}e^1-(0+0-\tfrac{1}{4}e^3)+10e^3-4e^3-(2e^1-4e^1)$$
$$=e^3(\tfrac{1}{2}+\tfrac{1}{4}+10-4)+e^1(\tfrac{1}{2}+\tfrac{1}{4}-2-4)$$
$$=\frac{27}{4}e^3-\frac{29}{4}e^1$$

$$\iint_R xe^y\,dA=\int_1^3\int_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} xe^y\,dx\,dy$$
$$=\int_1^3 \tfrac{1}{2}x^2e^y\Big]_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1}\,dy$$
$$=\int_1^3 \tfrac{1}{2}(2y-1)^2e^y-\tfrac{1}{2}(-\tfrac{1}{2}y+\tfrac{3}{2})^2e^y\,dy$$
$$=\int_1^3 \tfrac{1}{2}e^y((2y-1)^2-(\tfrac{3}{2}-\tfrac{y}{2})^2)\,dy$$
$$=\int_1^3 \tfrac{1}{2}e^y((2y-1)^2-\tfrac{1}{4}(3-y)^2)\,dy$$
$$=\frac{1}{8}\int_1^3 e^y(4(2y-1)^2-(3-y)^2)\,dy$$

$u=4(2y-1)^2-(3-y)^2$ $\quad dv=e^y$

$du=16(2y-1)-2(y-3)dy$ $\quad v=e^y$

$$=\frac{1}{8}[(4(2y-1)^2-(y-3)^2)e^y-\int_1^3(16(2y-1)-2(y-3))e^y\,dy]$$
$$=\frac{1}{8}[(4(2y-1)^2-(y-3)^2)e^y-\int_1^3(30y+22)e^y\,dy]$$

$u=30y+22$ $\quad dv=e^y$

$du=30dy$ $\quad v=e^y$

$$=\frac{1}{8}[(4(2y-1)^2-(y-3)^2)e^y-(30y+22)e^y+\int_1^3 e^y\cdot 30\,dy]$$
$$=\frac{1}{8}[(4(2y-1)^2-(y-3)^2)e^y-(30y+22)e^y+30e^y]\Big|_1^3$$
$$=\frac{1}{8}[e^y(4(2y-1)^2-(y-3)^2-(30y+22)+30)]\Big|_1^3$$
$$=\frac{1}{8}e^y(15x^2-40x+35)\Big|_1^3$$
$$=\frac{1}{4}(25e^3-5e)$$