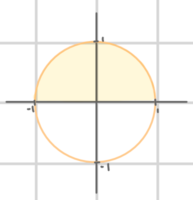
Standard 12: Double Integral-Polar Coordinates

polar coordinates

So far the region R could be described in terms of simple functions in Cartesian (Rectangular) coordinates. We now want to consider regions that are better described in terms of polar coordinates. For instance, finding the area of a section of a disk using polar coordinates would eliminate the square roots.

example. Use set builder notation to write out the region bounded by the northern hemisphere of the unit circle and the x-axis.



cartesian: $R=\{(x,y) \mid 0\le y\le \sqrt{1-x^2},\ -1\le x\le 1\}$

$=\{(x,y) \mid -\sqrt{1-y^2}\le x\le \sqrt{1-y^2},\ 0\le y\le 1\}$

polar: $R=\{(\theta,r) \mid 0\le\theta\le\pi,\ 0\le r\le 1\}$

$=\{(\theta,r) \mid 0\le r\le 1,\ 0\le\theta\le\pi\}$

Notice that the polar coordinates have constant limits of integration which makes it easier to compute.

It would be nice to convert our double integral into polar coordinates, but we can't just replace dx and dy with dr and dθ.
In the case of polar coordinates, $dA = r\,dr\,d\theta$. We will see why later when doing change of coordinates.

Cartesian ⟶ Polar:

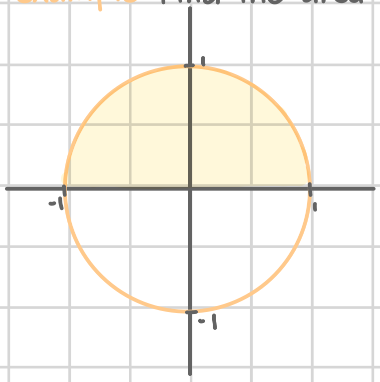
$x = r\cos(\theta)$

$y = r\sin(\theta)$

$x^2+y^2=r^2$ when not centered at the origin it may be a function of theta

$dA = r\,dr\,d\theta$ ex. $x^2+y^2=6y \Rightarrow r^2=6r\sin\theta \Rightarrow r=6\sin\theta$

example. Find the area bounded by the northern hemisphere of the unit circle and the x-axis using cartesian and polar coordinates.



cartesian: $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 1\,dy\,dx$

$=\int_{-1}^{1} y\Big|_{0}^{\sqrt{1-x^2}}\,dx$

$=\int_{-1}^{1} \sqrt{1-x^2}-0\,dx$

$=\frac{1}{2}\left(\arcsin(x)+x\sqrt{1-x^2}\right)\Big|_{-1}^{1}$

$=\frac{\pi}{2}$

polar: $\int_0^1\int_0^{\pi} r\,d\theta\,dr$

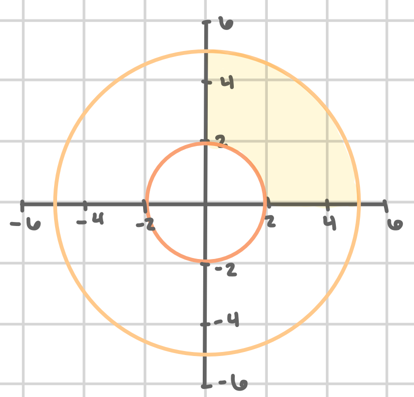
$=\int_0^1 r\theta\Big|_0^{\pi}\,dr$

$=\int_0^1 \pi r-0\cdot r\,dr$

$=\frac{1}{2}\pi r^2\Big|_0^1$

$=\frac{1}{2}\pi$

example. Compute $\iint_R 2xy\,dA$ where R is the region between the circle of radius 2 centered at the origin and the circle of radius 5 centered at the origin contained in the first quadrant.



$R=\{(x,y) \mid 2\le r\le\theta,\ 0\le\theta\le \pi/2\}$

$\iint_R 2xy\,dA = \int_0^{\pi/2}\int_2^5 2(r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$

$=\int_0^{\pi/2}\int_2^5 r^3\sin(2\theta)\,dr\,d\theta$ $\sin(2\theta)=2\sin\theta\cos\theta$

$=\int_0^{\pi/2}\frac{1}{4}r^4\sin(2\theta)\Big|_2^5\,d\theta$

$=\int_0^{\pi}\frac{1}{4}(609)\sin(2\theta)\,d\theta$

$=-\frac{609}{4}\cos(2\theta)\Big|_0^{\pi/2}$

$=\frac{609}{4}$

example. Set up, but do not solve, the double integral that gives the volume of the following region(s):

(i) bounded by $z=x^2+y^2$ and $z=2-\sqrt{x^2+y^2}$

$R=\{(x,y) \mid x^2+y^2-(2-\sqrt{x^2+y^2})\le 0\}$

$=\{(r,\theta) \mid r^2-2+r\le 0\}$

$=\{(r,\theta) \mid 0\le\theta\le 2\pi,\ 0\le r\le 1\}$

$V=\int_0^{2\pi}\int_0^1 (2-r)-(r^2)\,dr\,d\theta$

(ii) above xy-plane, below $y+z=1$, inside $x^2+y^2=1$

$R=\{(x,y) \mid x^2+y^2\le 1\}$

$=\{(r,\theta) \mid 0\le\theta\le 2\pi,\ 0\le r\le 1\}$

$\int_0^{2\pi}\int_0^1 (1-r\sin\theta)\cdot r\,dr\,d\theta$

(iii) $f(r,\theta)=1$, inside $r=3+\sin\theta$, outside $r=2$