Standard 13: Triple Integrals - Rectangular Coordinates

Triple Integrals

The next step up from integrating over a two-dimensional region is integrating over a three-dimensional region. It should come as no surprise that we use a triple integral to integrate over a three-dimensional region. The general triple integral is $\iiint_E f(x,y,z)\,dV$.

rectangular-prism region

Over the box $B = [a,b]\times[c,d]\times[r,s] = \{(x,y,z) \mid a\le x\le b,\ c\le y\le d,\ r\le z\le s\}$, the triple integral $\iiint_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$.

Note: we could rewrite this integral six different ways, dxdydz, dxdzdy, dydxdz, dydzdx, dzdxdy, dzdydx.

example. Evaluate $\iiint_B 8xyz\,dV$ for $B = [2,3]\times[1,2]\times[0,1]$.

$$\iiint_B 8xyz\,dxdydz = \int_2^3\int_1^2\int_0^1 8xyz\,dz\,dy\,dx$$
$$= \int_2^3\int_1^2 4xyz^2\Big|_0^1\,dy\,dx$$
$$= \int_2^3\int_1^2 4xy\,dy\,dx$$
$$= \int_2^3 2xy^2\Big|_1^2\,dx$$
$$= \int_2^3 2x(2)^2 - 2x(1)^2\,dx$$
$$= \int_2^3 6x\,dx$$
$$= 3x^2\Big|_2^3$$
$$= 3(3)^2 - 3(2)^2$$
$$= 27 - 12$$
$$= 15$$

$$= \int_1^2\int_2^3\int_0^1 8xyz\,dz\,dx\,dy$$
$$= \int_1^2\int_2^3 4xyz^2\Big|_0^1\,dx\,dy$$
$$= \int_1^2\int_2^3 4xy\,dx\,dy$$
$$= \int_1^2 2xy^2\Big|_2^3\,dy$$
$$= \int_1^2 2x(3)^2 - 2x(2)^2\,dy$$
$$= \int_1^2 18x - 8x\,dy$$
$$= \int_1^2 10x\,dy$$
$$= 5x^2\Big|_1^2$$
$$= 5(2)^2 - 5(1)^2$$
$$= 15$$

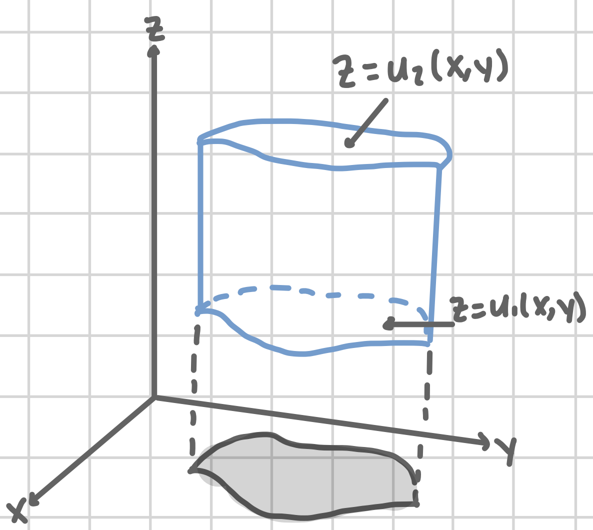
$$= \int_0^1\int_1^2\int_2^3 8xyz\,dx\,dy\,dz$$
$$= \int_0^1\int_1^2 4x^2yz\Big|_2^3\,dy\,dz$$
$$= \int_0^1\int_1^2 4(3)^2yz - 4(2)^2yz\,dy\,dz$$
$$= \int_0^1\int_1^2 36yz - 16yz\,dy\,dz$$
$$= \int_0^1\int_1^2 20yz\,dy\,dz$$
$$= \int_0^1 10y^2z\Big|_1^2\,dz$$
$$= \int_0^1 10(2)^2z - 10(1)^2z\,dz$$
$$= \int_0^1 30z\,dz$$
$$= 15z\Big|_0^1$$
$$= 15$$

Recall the calculus I explanation of the double integral: $A = \int_a^b g_2(x) - g_1(x)\,dx = \int_a^b\int_{g_1(x)}^{g_2(x)} 1\,dy\,dx$. We can make a similar assertion for triple integrals: $V = \iint_R u_2(x,y) - u_1(x,y)\,dA = \iint_R \int_{u_1(x,y)}^{u_2(x,y)} 1\,dz\,dA = \iiint_E 1\,dV$.

general region

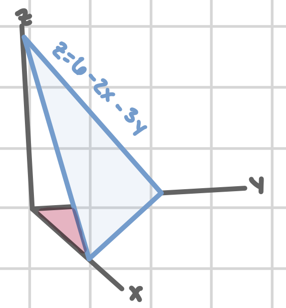
We have three ways to describe a general region in three-dimension:

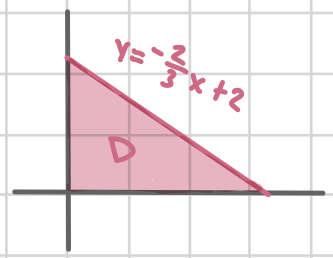
(i) $E = \{(x,y,z) \mid (x,y)\in D,\ u_1(x,y)\le z\le u_2(x,y)\}$



D is the shadow region in the xy-plane
$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$

example. Evaluate $\iiint_E 2x\,dV$ where E is the region under the plane $2x+3y+z=6$ that lies in the first octant.





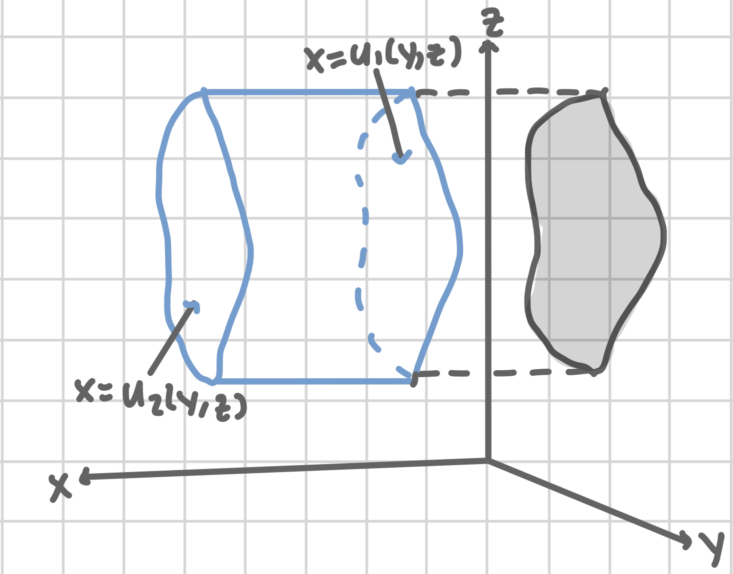
• Sketch 3-D graph
• Project onto xy-plane
• Find bounds for D and z

$D = \{(x,y) \mid 0\le x\le 3,\ 0\le y\le -\frac{2}{3}x+3\}$

$\iint_D f(x,y,z)\,dA = \int_0^3\int_0^{-\frac{2}{3}x+2} f(x,y,z)\,dy\,dx$

$\iiint_E 2x\,dV = \iint_D \left[\int_0^{6-2x-3y} 2x\,dz\right] dA$
$= \int_0^3\int_0^{-2/3x+2}\int_0^{6-2x-3y} 2x\,dz\,dy\,dx$

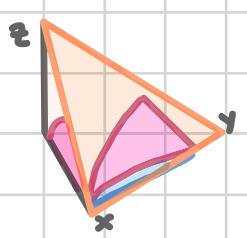
(ii) $E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$

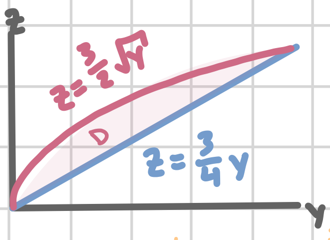


D is the shadow region in the yz-plane

$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$

example. Determine the volume of the region that lies behind the plane $x+y+z=8$ and in front of the region in the yz-plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}\cdot y$.





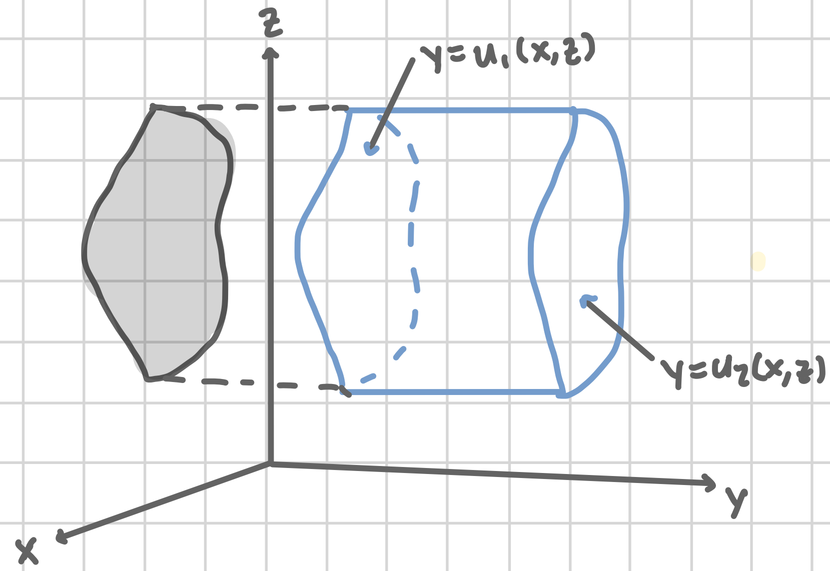
$D = \{(y,z) \mid 0 \le y \le 4,\ \frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}\}$

- Sketch 3-D graph
- Project on yz-plane
- Find bounds for D and x

$\iiint_E 1\,dV = \iint_D \left[\int_0^{8-y-z} 1\,dx\right] dA$

$= \int_0^4 \int_{3y/4}^{3\sqrt{y}/4} \int_0^{8-y-z} 1\,dx\,dz\,dy$

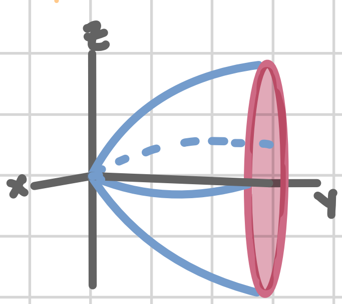
(iii) $E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$

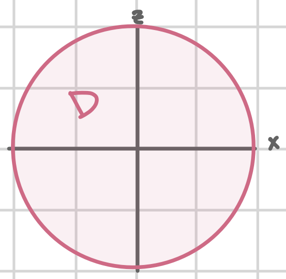


D is the shadow region in the xz-plane

$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\,]\,dA$

example. Evaluate $\iiint_E \sqrt{3x^2+3z^2}\,dV$ where E is the solid bounded by $y = 2x^2+2z^2$ and the plane $y=8$.





$D = \{(x,z) \mid x^2+z^2 \le 4\}$

$= \{(x,y) \mid -4 \le x \le 4,\ -\sqrt{4-x^2} \le z \le \sqrt{4-x^2}\}$

- Sketch 3-D graph
- Project onto xz-plane
- Find bounds for D and y

$\iiint_E \sqrt{3x^2+3z^2}\,dV = \iint_D \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\,dy\right] dA$

$= \int_{-4}^{4} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\,dy\,dz\,dx$

The last example highlights the fact that some regions are better described using slices of a circle than rectangles. As a thought exercise, imagine what the x,y,z bounds would be for a cone or a sphere and if they could be simplified with radii and angles.