Standard 15: Change of Variables

## Change of Variables

In Calculus I we used the u-substitution rule, $\int_a^b f(g(x))\cdot g'(x)\,dx = \int_c^d f(u)\,du$ where $u=g(x)$, in order to take an integral in terms of x's to an integral in terms of u's. We want to extend this idea to double and triple integrals. In fact we have already done this when we converted double integrals to polar and triple integrals to cylindrical and spherical. We just didn't cover the details of the conversions.

### transformations

Sometimes, like in the case of u-sub, the reason for the change in variable is to make the integral easier to compute. Other times we use change of variables to make the region into a nicer region. We call the equations that define the change of variables transformations. Typically, we start with a region R in xy-coordinates and transform it into a region S in uv-coordinates.

example. Determine the new region that we get by applying the given transformation to the region R.

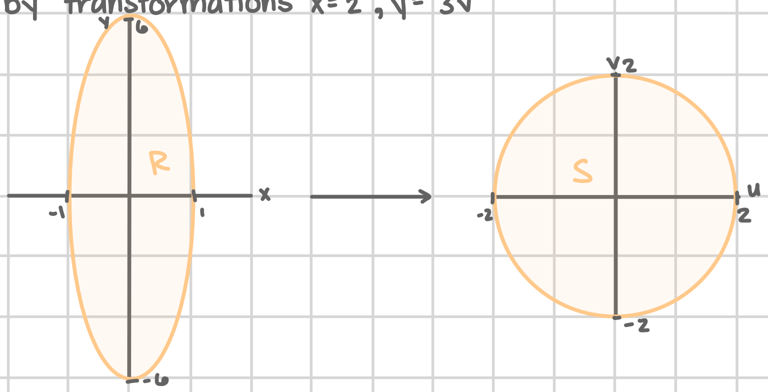
(i) $R=\{(x,y) \mid x^2+\frac{y^2}{36}\le 1\}$ by transformations $x=\frac{u}{2}$, $y=3v$

$x^2+\frac{1}{36}y^2=1$

$(\frac{u}{2})^2+\frac{1}{36}(3v)^2=1$

$\frac{1}{4}u^2+\frac{1}{4}v^2=1$

$u^2+v^2=4$




(ii) R is bounded by $y=-x+4$, $y=x+1$, and $y=\frac{1}{3}x-\frac{4}{3}$ by transformations $x=\frac{1}{2}(u+v)$, $y=\frac{1}{2}(u-v)$

$y=-x+4$

$\frac{1}{2}(u-v)=-\frac{1}{2}(u+v)+4$

$u-v=-u-v+8$

$2u=8$

$u=4$

$y=x+1$

$\frac{1}{2}(u-v)=\frac{1}{2}(u+v)+1$

$u-v=u+v+2$

$-2v=2$

$v=-1$

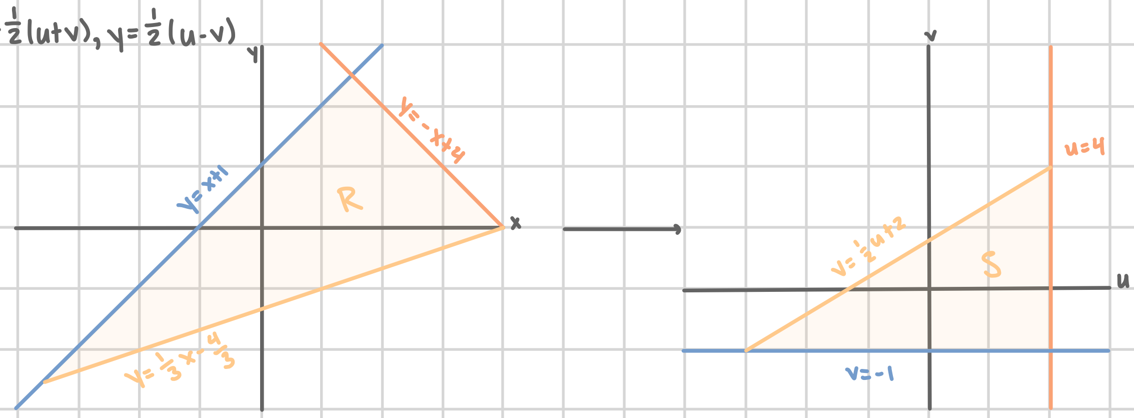
$y=\frac{1}{3}x-\frac{4}{3}$

$\frac{1}{2}(u-v)=\frac{1}{3}(\frac{1}{2}(u+v))-\frac{4}{3}$

$3u-3v=u+v-8$

$4v=2u+8$

$v=\frac{u}{2}+2$




### double integrals

In order to change variables in a double integral we need the Jacobian of the transformation. The Jacobian of the transformation $x=g(u,v)$, $y=h(u,v)$ is $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$

Suppose that we want to integrate $f(x,y)$ over the region R. Under the transformation $x=g(u,v)$, $y=h(u,v)$ the region becomes S and the integral becomes, $\iint_R f(x,y)\,dA=\iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|d\bar{A}$ where $d\bar{A}$ is just denoting the change to du, dv.

example. Show that when changing to polar coordinates we have $dA=r\,dr\,d\theta$.

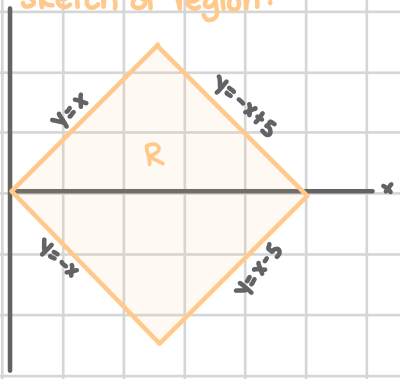
We have transformations $x=r\cos\theta$ and $y=r\sin\theta$.

The Jacobian for this transformation is $\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta-(-r\sin^2\theta)=r\cos^2\theta+r\sin^2\theta=r(\cos^2\theta+\sin^2\theta)=r$

Thus $dA=\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|dr\,d\theta=r\,dr\,d\theta$

example. Evaluate $\iint_R x+y\,dA$ where $R$ is the trapezoidal region with vertices given by $(0,0)$, $(5,0)$, $(5/2, 5/2)$, and $(5/2, -5/2)$ using transformation formulas $x = 2u+3v$ and $y = 2u-3v$.

Sketch of region:




Using x,y this would take two integrals. Let's do the transformation.

(i) $y = x$

$2u-3v = 2u+3v$

$6v = 0$

$v = 0$

(ii) $y = -x$

$2u-3v = -(2u+3v)$

$4u = 0$

$u = 0$

(iii) $y = -x+5$

$2u-3v = -(2u+3v)+5$

$4u = 5$

$u = 5/4$

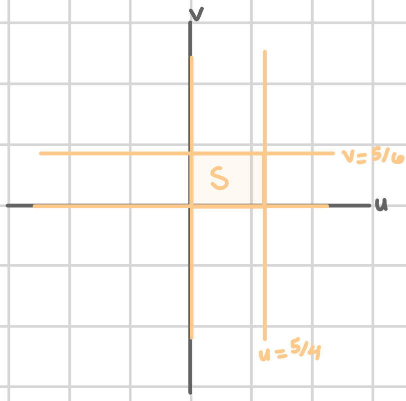
(iv) $y = x-5$

$2u-3v = 2u+3v-5$

$-6v = -5$

$v = 5/6$

The region $S$ is a rectangle with sides $u=0$, $v=0$, $u=5/4$, $v=5/6$, i.e. $0 \le u \le 5/4$ and $0 \le v \le 5/6$




The Jacobian is $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -6-6 = -12.$

So the integral is
$$\iint_R x+y\,dA = \int_0^{5/6}\int_0^{5/4} ((2u+3v)+(2u-3v))\,|-12|\,dA$$
$$= \int_0^{5/6}\int_0^{5/4} 48u\,d\bar{A}$$
$$= \int_0^{5/6} 24u^2 \Big]_0^{5/4} dv$$
$$= \int_0^{5/6} \frac{75}{2}\,dv$$
$$= \frac{75}{2}\Big]_0^{5/6}$$
$$= \frac{125}{4}$$

## triple integrals

Triple integrals have an additional variable at each step. We start with a region $E$ and use transformations $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$ to transform the region to a new region $F$. We still need the Jacobian of the transformations:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial z}{\partial v}\right) - \frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial z}{\partial u}\right) + \frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)$$

The integral under this transformation is $\iiint_E f(x,y,z)\,dV = \iiint_F f(g(u,v,w), h(u,v,w), k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| d\bar{V}$ where $d\bar{V}$ is just denoting the change of variables.

example. Verify that $dV = \rho^2\sin(\varphi)\,d\rho\,d\theta\,d\varphi$ when using spherical coordinates.

We have transformations $x = \rho\sin(\varphi)\cos\theta$, $y = \rho\sin(\varphi)\sin(\theta)$, $z = \rho\cos(\varphi)$.

The Jacobian for this transformation is
$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \sin(\varphi)\cos(\theta) & -\rho\sin(\varphi)\sin(\theta) & \rho\cos(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) & \rho\sin(\varphi)\cos(\theta) & \rho\cos(\varphi)\sin(\theta) \\ \cos(\varphi) & 0 & -\rho\sin(\varphi) \end{vmatrix}$$

$$= \rho^2\sin^3(\varphi)\cos^2(\theta) - \rho^2\sin(\varphi)\cos^2(\varphi)\sin^2(\theta) + 0 - \rho^2\sin^3(\varphi)\sin^2(\theta) - 0 - \rho^2\sin(\varphi)\cos^2(\varphi)\cos^2(\theta)$$
$$= -\rho^2\sin(\varphi)(\sin^2(\varphi) + \cos^2(\varphi))$$
$$= -\rho^2\sin(\varphi)$$

Thus $dV = |-\rho^2\sin(\varphi)|\,d\rho\,d\theta\,d\varphi = \rho^2\sin(\varphi)\,d\rho\,d\theta\,d\varphi$