Standard 16: Line Integrals

Line Integrals

Vector Fields

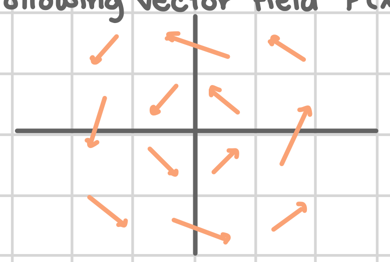
A vector field on two (or three) dimensional space is a function $\vec{F}$ that assigns to each point (x,y) (or (x,y,z)) a two (or three) dimensional vector given by $\vec{F}(x,y)$ (or $\vec{F}(x,y,z)$). You might have seen these in physics to show the flow of a fluid or wind movement in the air.
standard notation: $\vec{F}(x,y) = P(x,y)\,\hat{\imath} + Q(x,y)\,\hat{\jmath}$ (or $\vec{F}(x,y,z) = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\vec{k}$) where P, Q (and R) are scalar functions.

example. Sketch the following vector field $\vec{F}(x,y) = -y\,\hat{\imath} + x\,\hat{\jmath}$ sample evaluations:

$\vec{F}(\frac{1}{2},\frac{1}{2}) = -\frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$

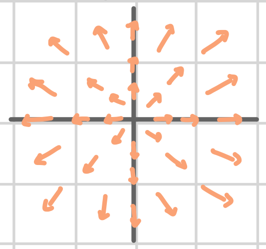
$\vec{F}(\frac{1}{2},-\frac{1}{2}) = \frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$

$\vec{F}(\frac{3}{2},\frac{1}{4}) = -\frac{1}{4}\hat{\imath} + \frac{3}{2}\hat{\jmath}$



example. Find the gradient vector field of the function $f(x,y) = x^2 + y^2$.

$\vec{F}(x,y) = \nabla f(x,y) = 2x\,\hat{\imath} + 2y\,\hat{\jmath}$



Line Integrals (with respect to arc length)

In calculus I, we integrated $f(x)$, a function of a single variable, over an interval $[a,b]$, i.e. x takes on the values of the line segment from a to b. With line integrals we want to integrate the function $f(x,y)$, a function of two variables, over the curve C, i.e. the values (x,y) must lie on the curve C. Note that this is different from double integrals where the values (x,y) came out of a 2D region.

Given a curve C parameterized by $x = h(t)$, $y = g(t)$ with $a \le t \le b$ (also written $\vec{r}(t) = h(t)\hat{\imath} + g(t)\hat{\jmath}$ for $a \le t \le b$). The line integral of $f(x,y)$ along C is denoted by $\int_C f(x,y)\,ds$ where ds is denoting that we are going over a curve (rather than area being dA).
Recall from arc length $L = \int_a^b ds$ where $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$
We use this to compute the line integral $\int_C f(x,y)\,ds = \int_a^b f(h(t),g(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt = \int_a^b f(h(t),g(t))\,\|\vec{r}\,'(t)\|\,dt$.

example. Evaluate $\int_C xy^4\,ds$ where C is the right half of the circle, $x^2+y^2=16$ traced counter clockwise.
The parameterization of $x^2+y^2=16$ is given by $x = 4\cos t$, $y = 4\sin t$ and the right half of the circle comes from $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.
Now compute $ds = \|r'(t)\| = \sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt = 4\,dt$. Which gives the line integral $\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} 4\cos t\,(4\sin t)^4\,(4)\,dt = \frac{8192}{5}$.

example. Find the arclength of the curve parameterized by $r(t) = \langle 4\cos(t), 4\sin(t)\rangle$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

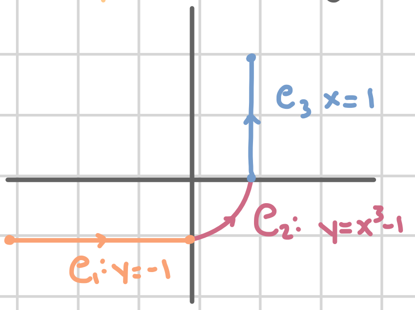
$$\begin{aligned} L = \int_a^b ds &= \int_a^b \|\vec{r}\,'(t)\|\,dt \\ &= \int_{-\pi/2}^{\pi/2} 2\,dt \\ &= 2t\,\Big|_{-\pi/2}^{\pi/2} \\ &= 2\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}\right) \\ &= 2\pi \end{aligned}$$

$$\vec{r}\,'(t) = \langle -4\sin(t), 4\cos(t)\rangle$$

$$\begin{aligned} \|\vec{r}\,'(t)\| &= \sqrt{(-4\sin(t))^2 + (4\cos(t))^2} \\ &= \sqrt{4\sin^2(t) + 4\cos^2(t)} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

This line integral utilizes the fact that C is a smooth curve i.e. continuous and $\vec{r}\,'(t) \neq 0$ for all t. We now have to consider piecewise smooth curve, i.e. C can be written as the union of a finite collection of smooth curves $C_1, \ldots, C_n$ where the endpoint of C_i is the starting point of C_{i+1}. The line integral of the piecewise smooth curve $C = \bigcup_{i=1}^{n} C_i$ is $\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \ldots + \int_{C_n} f(x,y)\,ds$.

example. Evaluate $\int_C 4x^3 ds$ where C is the curve shown below:



parameterize:

C_1: $x=t$, $y=-1$ $\quad -2 \le t \le 0$

C_2: $x=t$, $y=t^3-1$ $\quad 0 \le t \le 1$

C_3: $x=1$, $y=t$ $\quad 0 \le t \le 2$

$C = C_1 \cup C_2 \cup C_3$

$\int_{C_1} 4x^3 ds = \int_{-2}^{0} 4t^3 \sqrt{(1)^2+(0)^2} = -16$

$\int_{C_2} 4x^3 ds = \int_0^1 4t^3 \sqrt{1+9t^4} = \frac{2}{27}(10^{3/2}-1) = 2.268$

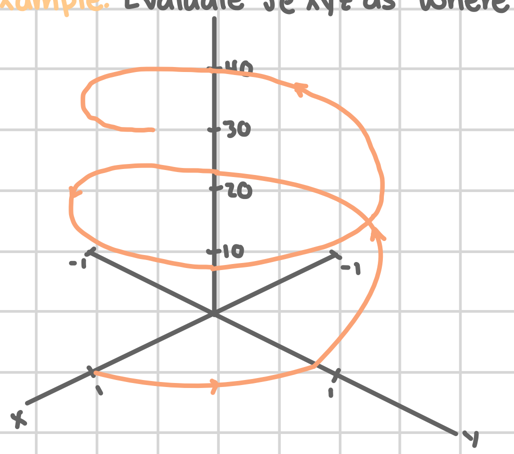
$\int_{C_3} 4x^3 ds = \int_0^2 4(1)^3 \sqrt{(0)^2+(1)^2}\, dt = 8$

$\int_C 4x^3 ds = \int_{C_1} 4x^3 ds + \int_{C_2} 4x^3 ds + \int_{C_3} 4x^3 ds = -5.732$

For sake of completion we include a 3-dimensional example:

example. Evaluate $\int_C xyz\, ds$ where C is the helix given by $\vec{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$ and $0 \le t \le 4\pi$.

$\int_C xyz\, ds = \int_0^{4\pi} 3t \cos(t)\sin(t) \sqrt{\sin^2 t + \cos^2 t + 9}\, dt$



Line Integrals (with respect to x and/or y)

The previous section covered line integrals with respect to arclength. This section will look at line integrals with respect to x and/or y.

We start with a parameterization of a two-dimensional curve C: $x = x(t)$, $y = y(t)$, $a \le t \le b$.

- The line integral of f with respect to x is $\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\, x'(t)\, dt$.
- The line integral of f with respect to y is $\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\, y'(t)\, dt$.

You may also be asked to find the combination of these:

- $\int_C P dx + Q dy = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$

example. Evaluate $\int_C \sin(\pi y)\,dy + yx^2\,dx$ where C is the line segment from $(1,4)$ to $(0,2)$.

parameterize: $\vec{r}(t) = (1-t)\langle 1,4 \rangle + t\langle 0,2 \rangle = \langle 1-t, 4-2t \rangle$ for $0 \le t \le 1$

$$\begin{aligned}
\int_C \sin(\pi y)\,dy + yx^2\,dx &= \int_C \sin(\pi y)\,dy + \int_C yx^2\,dx \\
&= \int_0^1 \sin(\pi(4-2t))(-2)\,dt + \int_0^1 (4-2t)(1-t)^2(-1)\,dt \\
&= -\tfrac{1}{\pi}\cos(4\pi - 2\pi t)\Big]_0^1 - \left(-\tfrac{1}{2}t^4 + \tfrac{8}{3}t^3 - 5t^2 + 4t\right)\Big]_0^1 \\
&= -7/6
\end{aligned}$$

In three-dimensions, $\int_C P dx + Q dy + R dz = \int_C P(x,y,z)\,dx + \int Q(x,y,z)\,dy + \int_C R(x,y,z)\,dz$

example. Evaluate $\int_C y\,dx + x\,dy + z\,dz$ where C is given by $x = \cos t$, $y = \sin t$, $z = t^2$, $0 \le t \le 2\pi$.

$$\begin{aligned}
\int_C y\,dx + x\,dy + z\,dz &= \int_C y\,dx + \int_C x\,dy + \int_C z\,dz \\
&= \int_0^{2\pi} \sin t(-\sin t)\,dt + \int_0^{2\pi} \cos t(\cos t)\,dt + \int_0^{2\pi} t^2(2t)\,dt \\
&= -\int_0^{2\pi} \sin^2 t\,dt + \int_0^{2\pi} \cos^2 t\,dt + \int_0^{2\pi} 2t^3\,dt \\
&= -\tfrac{1}{2}\int_0^{2\pi}(1-\cos(2t))\,dt + \tfrac{1}{2}\int_0^{2\pi}(1+\cos(2t))\,dt + \int_0^{2\pi} 2t^3\,dt \\
&= \left(-\tfrac{1}{2}\left(t - \tfrac{1}{2}\sin(2t)\right) + \tfrac{1}{2}\left(t + \tfrac{1}{2}\sin(2t)\right) + \tfrac{1}{2}t^4\right)\Big]_0^{2\pi} \\
&= 8\pi^4
\end{aligned}$$

Line Integrals of Vector Fields

We start with the vector field $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$ and the three-dimension, smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ $a \le t \le b$. The line integral of $\vec{F}$ along C is $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$.

example. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = 8x^2yz\,\vec{i} + 5z\,\vec{j} - 4xy\,\vec{k}$ and C is the curve given by $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^3\,\vec{k}$, $0 \le t \le 1$.

$\vec{F}(\vec{r}(t)) = 8t^2(t^2)(t^3)\,\vec{i} + 5t^3\,\vec{j} - 4t(t^2)\,\vec{k} = 8t^7\,\vec{i} + 5t^3\,\vec{j} - 4t^3\,\vec{k}$

$\vec{r}\,'(t) = \vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 8t^7 + 10t^4 - 12t^5$

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^1 8t^7 + 10t^4 - 12t^5\,dt \\ &= \left(t^8 + 2t^5 - 2t^6\right)\Big]_0^1 \\ &= 1.\end{aligned}$$

We can also rewrite $\int_C \vec{F}\cdot d\vec{r}$ using the previous section:

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_a^b (P\vec{i} + Q\vec{j} + R\vec{k})\cdot(x'\vec{i} + y'\vec{j} + z'\vec{k})\,dt \\ &= \int_a^b (Px' + Qy' + Rz')\,dt \\ &= \int_a^b Px'\,dt + \int_a^b Qy'\,dt + \int_a^b Rz'\,dt \\ &= \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz \\ &= \int_C P\,dx + Q\,dy + R\,dz.\end{aligned}$$

work

One application of line integrals of vector fields is work. Suppose we have a particle moving along a path C in the presence of a force field $\vec{F}$. The work performed is given by $W = \int_C \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\cdot\|\vec{r}\,'(t)\|\,dt = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}$.

example. Find the work done by the force field $\vec{F}(x) = \langle xy, 3y^2\rangle$ and C is parameterized by $\vec{r}(t) = \langle 11t^4, t^3\rangle$, $0 \le t \le 1$.

$$\begin{aligned}W = \int_C \vec{F}\cdot d\vec{r} &= \int_0^b \vec{F}(\vec{r}(t))\,\vec{r}\,'(t)\,dt \\ &= \int_0^1 484t^{10} + 9t^8\,dt \\ &= 44t^{11} + t^9\Big|_0^1 \\ &= 44 + 1 - (0+0) \\ &= 45\end{aligned}$$

$\vec{r}\,'(t) = \langle 44t^3, 3t^2\rangle$

$$\begin{aligned}\vec{F}(\vec{r}(t)) &= \langle (11t^4)(t^3), 3(t^3)^2\rangle \\ &= \langle 11t^7, 3t^6\rangle\end{aligned}$$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 484t^{10} + 9t^8$

Additional Formulas

arc length: $\int_{-C} f\,ds = \int_C f\,ds$

vector: $\int_{-C} f\,dx = -\int_C f\,dx$

$\int_{-C} f\,dy = -\int_C f\,dy$

$\int_{-C} f\,dz = -\int_C f\,dz$

$\int_{-C} P\,dx + Q\,dy + R\,dz = -\int_C P\,dx + Q\,dy + R\,dz$