Standard 17: Fundamental Theorem of Line Integrals

Fundamental Theorem for Line Integrals

Recall from Calculus I the Fundamental Theorem of Calculus, $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Fundamental Theorem for Line Integrals

There is a version of this for certain vector fields, Suppose that C is a smooth curve given by $\vec{r}(t)$, $a \le t \le b$ and that f is a function whose gradient vector, ∇f, is continuous on $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ where $\vec{r}(a)$ is the intial point on C and $\vec{r}(b)$ is the final point on C.

proof. $\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_a^b \left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}\right) dt = \int_a^b \frac{d}{dt}[f(\vec{r}(t))]\,dt \overset{FTC}{=} f(\vec{r}(b)) - f(\vec{r}(a))$. #

example. Evaluate $\int_C \nabla f \cdot d\vec{r}$ where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $(1, 1/2, 2)$ and ends at $(2,1,-1)$.

note: the specified path is not given, but it is not necessary for this problem only $\vec{r}(a)$ and $\vec{r}(b)$ is needed.

$\int_C \nabla f \cdot d\vec{r} = f(2,1,-1) - f(1, 1/2, 2) = \cos(2\pi) + \sin\pi - 2(1)(-1) - (\cos\pi + \sin(\pi/2) - 1(1/2)(2)) = 4.$

Definitions

Suppose that $\vec{F}$ is a continuous vector field in some domain D.

1. $\vec{F}$ is a conservative vector field if there is a function f such that $\vec{F} = \nabla f$.
2. The function f for a conservative vector field is called a potential function for the vector field.
3. $\int_C \vec{F} \cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 and C_2 in D with the same intial and final points.
4. A path C is called closed if its initial and final points are the same point.
5. A path C is simple if it doesn't cross itself.
6. A region D is open if it doesn't contain any of its boundary points.
7. A region D is connected if we can connect any two points in the region with a path that lies completely in D.
8. A region is simply-connected if it is connected and it contains no holes

Facts

1. $\int_C \nabla f \cdot d\vec{r}$ is independent of path
2. If $\vec{F}$ is a conservative vector field then $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.
3. If $\vec{F}$ is a continuous vector field on an open connected region D and if $\int_C \vec{F} \cdot d\vec{r}$ is independent of path then $\vec{F}$ is a conservative v.f.
4. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path then $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C.
5. If $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C then $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

Conservative Vector Fields & Potential Functions

Fact 2 gives us an easy way to evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ using a potential function for $\vec{F}$.

theorem. Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open and simply-connected region D. Then if P and Q have continuous first order partial derivatives in D and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, the vector field $\vec{F}$ is conservative.

example. Determine if the following vector fields are conservative or not.

(a) $\vec{F}(x,y) = (x^2 - yx)\vec{i} + (y^2 - xy)\vec{j}$

$P = x^2 - yx$ and $Q = y^2 - xy$ $\Rightarrow$ $\frac{\partial P}{\partial y} = -x$ and $\frac{\partial Q}{\partial x} = -y$ $\Rightarrow$ $\vec{F}$ is not conservative.

(b) $\vec{F}(x,y) = (2xe^{xy} + x^2ye^{xy})\vec{i} + (x^3e^{xy} + 2y)\vec{j}$

$P = 2xe^{xy} + x^2ye^{xy}$ and $Q = x^3e^{xy} + 2y$ $\Rightarrow$ $\frac{\partial P}{\partial y} = 2x^2e^{xy} + x^2e^{xy} + x^3ye^{xy} = 3x^2e^{xy} + x^3ye^{xy}$ and $\frac{\partial Q}{\partial x} = 3x^2e^{xy} + x^3ye^{xy}$ $\Rightarrow$ $\vec{F}$ is conservative.

If $\vec{F}$ is a conservative function then a potential function exists. The potential function is f s.t. $\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = P\vec{i} + Q\vec{j} = \vec{F}$

This is equivalent to $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$. Which we can solve by integrating: $f(x,y) = \int P(x,y)\,dx$ or $f(x,y) = \int Q(x,y)\,dy$.

examples. Determine if the following vector fields are conservative and if so, find a potential function for the vector field.

(a) $\vec{F} = (2x^3y^4 + x)\vec{i} + (2x^4y^3 + y)\vec{j}$

$P = 2x^3y^4 + x = \frac{\partial f}{\partial x} \Rightarrow \frac{\partial P}{\partial y} = 8x^3y^3$

$Q = 2x^4y^3 + y \quad \frac{\partial f}{\partial y} \Rightarrow \frac{\partial Q}{\partial x} = 8x^3y^3$

it is conservative

$f(x,y) = \int P(x,y)\,dx$
$= \int (2x^3y^4 + x)\,dx$
$= \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + h(y)$

$\frac{\partial f}{\partial y} = 2x^4y^3 + h'(y) = 2x^4y^3 + y = Q$

so $h'(y) = y$

$h(y) = \int h'(y)\,dy = \int y\,dy = \frac{1}{2}y^2 + c$

so $f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$

OR

$f(x,y) = \int Q(x,y)\,dy$
$= \int (2x^4y^3 + y)\,dy$
$= \frac{1}{2}x^4y^4 + \frac{1}{2}y^2 + h(x)$

$\frac{\partial f}{\partial x} = 2x^3y^4 + h'(x) = 2x^3y^4 + x = P$

so $h'(x) = x$

$h(x) = \int h'(x)\,dx = \int x\,dx = \frac{1}{2}x^2 + c$

so $f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}y^2 + \frac{1}{2}x^2 + c$

verify by $\nabla f = \langle 2x^3y^4 + x, 2x^4y^3 + y \rangle = \vec{F}$

(b) $\vec{F}(x,y) = (2xe^{xy} + x^2ye^{xy})\vec{i} + (x^3e^{xy} + 2y)\vec{j}$

see example 1 for verification of conservative

$\frac{\partial f}{\partial x} = 2xe^{xy} + x^2ye^{xy}$ and $\frac{\partial f}{\partial y} = x^3e^{xy} + 2y$

$f(x,y) = \int 2xe^{xy} + x^2ye^{xy}$ OR $f(x,y) = \int x^3e^{xy} + 2y\,dy$

use second one: $f(x,y) = x^2e^{xy} + y^2 + h(x)$

$\frac{\partial f}{\partial x} = 2xe^{xy} + x^2ye^{xy} + h'(x) = 2xe^{xy} + x^2ye^{xy} = P$

$h'(x) = 0 \Rightarrow h(x) = c$

$f(x,y) = x^2e^{xy} + y^2 + c$

We have no way to determine if a 3-dimensional vector field is conservative, but given a 3D conservative v.f., we can find a potential function.

$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = P\vec{i} + Q\vec{j} + R\vec{k} = \vec{F}$

example. Find a potential function for the vector field $\vec{F} = (2x\cos(y) - 2z^3)\vec{i} + (3 + 2ye^z - x^2\sin(y))\vec{j} + (y^2e^z - 6xz^2)\vec{k}$.

$\frac{\partial f}{\partial x} = 2x\cos(y) - 2z^3 \qquad \frac{\partial f}{\partial y} = 3 + 2ye^z - x^2\sin(y) \qquad \frac{\partial f}{\partial z} = y^2e^z - 6xz^2 \leftarrow$ use this one

$f(x,y,z) = \int (y^2e^z - 6xz^2)\,dz = y^2e^z - 2xz^3 + g(x,y)$ integrate with respect to z

$\frac{\partial f}{\partial x} = -2z^3 + g_x(x,y) = 2x\cos(y) - 2z^3 = P$ differentiate w/ respect to x & set equal to P

$g_x(x,y) = 2x\cos(y) \Rightarrow g(x,y) = x^2\cos(y) + h(y)$ integrate partial

$f(x,y,z) = y^2e^z - 2xz^3 + x^2\cos(y) + h(y)$ plug g(x,y) into f(x,y)

$\frac{\partial f}{\partial y} = 2ye^z - x^2\sin(y) + h'(y) = 3 + 2ye^z - x^2\sin(y) = Q$ integrate with respect to y & set equal to Q

$h'(y) = 3 \Rightarrow h(y) = 3y + c$ integrate partial

$f(x,y,z) = y^2e^z - 2xz^3 + x^2\cos(y) + 3y + c$ plug h(y) into f(x,y,z)

example. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (2x^3y^4 + x)\vec{i} + (2x^4y^3 + y)\vec{j}$ and C is given by $\vec{r}(t) = (t\cos(\pi t) - 1)\vec{i} + \sin(\frac{\pi t}{2})\vec{j}$, $0 \le t \le 1$.

From example 3a, this vector field is conservative and the potential function is $f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$.

This integral is path independent so we can use $\int_C \vec{F} \cdot d\vec{r} = \int \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$ where $\vec{r}(1) = \langle -2, 1 \rangle$ and $\vec{r}(0) = \langle -1, 0 \rangle$.

So $\int_C \vec{F} \cdot d\vec{r} = f(-2,1) - f(-1,0) = (\frac{21}{2} + c) - (\frac{1}{2} + c) = 10$