Standard 18: Curl and Divergence

Curl and Divergence

curl

Given the vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ the <u>curl</u> is defined to be $\text{curl}(\vec{F}) = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$.

There is an alternative (potentially easier) definition of the curl of a vector field, but it takes some set up. First we define the ∇ (pronounced del) operator, $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$. We have used it previously when taking the gradient of a function ($\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$). We can use ∇ to define the curl as $\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

Facts

1. If $f(x,y,z)$ has continuous second order partial derivatives then $\text{curl}(\nabla f) = \vec{0}$.
2. If $\vec{F}$ is a conservative vector field then $\text{curl}\,\vec{F} = \vec{0}$.
3. If $\vec{F}$ is defined on all of $\mathbb{R}^3$ whose components have continuous first order partial derivative and $\text{curl}(\vec{F}) = \vec{0}$ then $\vec{F}$ is a conservative vector field

example. Determine if $\vec{F} = xy^2z^3\vec{i} + x^3yz^2\vec{j} + x^2y^3z\vec{k}$ is a conservative vector field.

fact 3 (or 2) tells us that all we need to do is compute the curl and see if we get the zero vector or not.

$$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & x^3yz^2 & x^2y^3z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}x^2y^3z - \frac{\partial}{\partial z}x^3yz^2\right)\vec{i} - \left(\frac{\partial}{\partial x}x^2y^3z + \frac{\partial}{\partial z}xy^2z^3\right)\vec{j} + \left(\frac{\partial}{\partial x}x^3yz^2 - \frac{\partial}{\partial y}xy^2z^3\right)\vec{k}$$

$$= \langle 3x^2y^2z - 2x^3yz, -2xy^3z + 3xy^2z^2, 3x^2yz^2 - 2xyz^3 \rangle$$

$$\neq 0$$

thus $\vec{F}$ is not conservative.

divergence

Given the vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ the <u>divergence</u> is defined to be $\text{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. This can be simplified by the ∇ operator to be $\text{div}(\vec{F}) = \nabla \cdot \vec{F}$.

example. Compute $\text{div}(\vec{F})$ of $\vec{F} = xy^2z^3\vec{i} + x^3yz^2\vec{j} + x^2y^3z\vec{k}$.

$$\text{div}(\vec{F}) = \frac{\partial}{\partial x}(xy^2z^3) + \frac{\partial}{\partial y}(x^3yz^2) + \frac{\partial}{\partial z}(x^2y^3z)$$

$$= y^2z^3 + x^3z^2 + x^2y^3$$

Fact

1. For any $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, $\text{div}(\text{curl}\,\vec{F}) = 0$