Standard 19: Green's Theorem

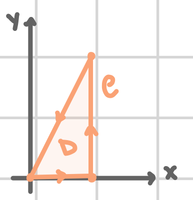
Green's Theorem

theorem. Let C be a positively oriented, piecewise smooth, simple closed curve and let D be the region enclosed by the curve. If P and Q have continuous first order partial derivatives on D then, $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$.

notation. line integrals under such curve conditions are sometimes written $\oint P\,dx + Q\,dy$.

example. Use Green's Theorem to evaluate $\oint_C xy\,dx + x^2y^3\,dy$ where C is the triangle with vertices $(0,0), (1,0), (1,2)$ with positive orientation.

Let's sketch the curve C and the enclosed region D:



$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2x\}$

$P = xy$ and $Q = x^2y^3$

So using Green's theorem: $\oint_C xy\,dx + x^2y^3\,dy = \iint_D 2xy^3 - x\,dA$

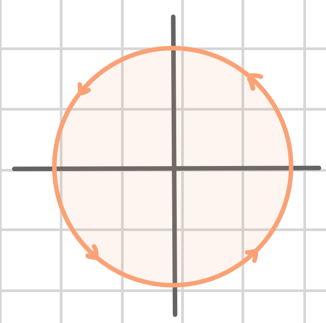
$$= \int_0^1\int_0^{2x} 2xy^3 - x\,dy\,dx$$
$$= \int_0^1 \left(\tfrac{1}{2}xy^4 - xy\right)\Big]_0^{2x} dx$$
$$= \int_0^1 8x^5 - 2x^2\,dx$$
$$= \left(\tfrac{4}{3}x^6 - \tfrac{2}{3}x^3\right)\Big]_0^1 = \tfrac{2}{3}$$

example. Evaluate $\oint_C y^3dx - x^3dy$ where C is the positively oriented circle of radius 2 centered at the origin.

A circle will satisfy the conditions of Green's Theorem since it is closed and simple. $P = y^3$ and $Q = -x^3$.

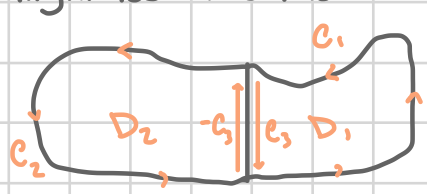
Using Green's Theorem: $\oint_C y^3dx - x^3dy = \iint_D -3x^2 - 3y^2\,dA$

$$= -3\int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta$$
$$= -3\int_0^{2\pi} \tfrac{1}{4}r^4\Big]_0^2\,d\theta$$
$$= -3\int_0^{2\pi} 4\,d\theta$$
$$= -3(4\theta)\Big]_0^{2\pi}$$
$$= -24\pi$$



Regions with Holes

Green's theorem will not work on regions that have holes in them, but many regions do have holes, so we must find a loophole to fix this. Let's start the process by trying it on a space without holes first. Let $D = D_1 \cup D_2$ be the union of two regions with boundary of D_1 being $C_1 \cup C_3$ and the boundary of D_2 being $C_2 \cup (-C_3)$. Then the boundary of D would be $C = (C_1 \cup C_3) \cup ((-C_3) \cup C_2) = C_1 \cup C_2$ and the region might look like this:



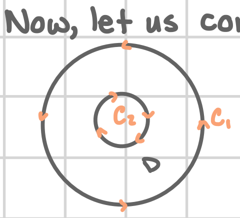
Starting with basic double integral laws: $\iint_D (Q_x - P_y)\,dA = \iint_{D_1 \cup D_2} (Q_x - P_y)\,dA = \iint_{D_1} (Q_x - P_y)\,dA + \iint_{D_2} (Q_x - P_y)\,dA$

We use Green's integral on each part:

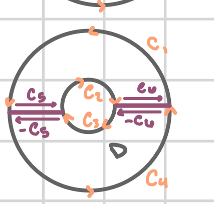
$$= \oint_{C_1 \cup C_3} P\,dx + Q\,dy + \oint_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \oint_{C_1} P\,dx + Q\,dy + \oint_{C_3} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy + \oint_{-C_3} P\,dx + Q\,dy$$

Therefore $\iint_D (Q_x - P_y)\,dA = \oint_{C_1 \cup C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy$.

Now, let us consider what this tells us on a washer:



In its current state, the region D has a hole in it so we are unable to use Green's Theorem with the curve $C = C_1 \cup C_2$. But, if we remove the hole by cutting the disk in half to make a new picture then we get the following sketch:



Which has region D_1 with boundary $C_1 \cup C_2 \cup C_5 \cup C_6$ and region D with boundary $C_3 \cup C_4 \cup (-C_5) \cup (-C_6)$ and we can use Green's Theorem on each of the parts. Going through the same calculation as above, we receive:

$\iint_D (Q_x - P_y)\,dA) = \oint_{C_1 \cup C_2 \cup C_3 \cup C_4} P\,dx + Q\,dy = \oint_C P\,dx + Q\,dy$.

example. Evaluate $\oint_C y^3dx - x^3dy$ where C are the two circles of radius 2 and radius 1 centered at the origin with positive orientation.

$\oint_C y^3dx - x^3dy = -3\iint_D (x^2 + y^2)\,dA$

$$= -3\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta$$
$$= -3\int_0^{2\pi} \left[\tfrac{1}{4}r^4\right]_1^2 d\theta$$
$$= -3\int_0^{2\pi} \tfrac{15}{4}\,d\theta$$
$$= -3\left[\tfrac{15}{4}\theta\right]_0^{2\pi}$$
$$= -\tfrac{45\pi}{2}$$

Additional formula:

$A = \iint_D 1\,dA$ so $Q_x - P_y = 1$

$\Rightarrow A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint x\,dy - y\,dx$

where C is the boundary of D