Standard 20: Surface Integrals

Parametric Surfaces

Recall how we parameterized a curve using values of t in some interval plugged into:
$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ which resulted in position vectors for points on the
With surfaces, we take values (u,v) in some 2D space D and plug them into:
$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$ which results in position vectors for points on the surface.
This $\vec{r}(u,v)$ is called the parametric representation of the parametric surface S.

example. Give parametric representation for each of the following surfaces.

(a) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$
Notice that the equation is already in the form $x = f(y,z)$ so we can pick $y=y, z=z$ and $x = 5y^2 + 2z^2 - 10$ which gives: $\vec{r}(y,z) = (5y^2+2z^2-10)\vec{i} + y\vec{j} + z\vec{k}$.

(b) The elliptic paraboloid $x = 5y^2+2z^2-10$ in front of the yz-plane
simply keep the parameterization from part (a): $\vec{r}(y,z) = 5y^2 + 2z^2 - 10$ and add the requirement $x \geq 0$ which becomes $5y^2 + 2z^2 \geq 10$

(c) The sphere $x^2+y^2+z^2 = 30$
Using spherical coordinates as inspiration take $\rho = \sqrt{30}$ then $x = \rho\sin\varphi\cos\theta$, $z = \rho\cos\varphi$, $y = \rho\sin\varphi\sin\theta$.

$z = f(x,y) \Rightarrow \vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$
$x = f(y,z) \Rightarrow \vec{r}(y,z) = f(y,z)\vec{i} + y\vec{j} + z\vec{k}$
$y = f(x,z) \Rightarrow \vec{r}(x,z) = x\vec{i} + f(x,z)\vec{j} + z\vec{k}$

We can use this parametric representation to find the tangent plane to the surface:
Given $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$, define $\vec{r}_u(u,v) = \frac{\partial x}{\partial u}(u,v)\vec{i} + \frac{\partial y}{\partial u}(u,v)\vec{j} + \frac{\partial z}{\partial u}(u,v)\vec{k}$ and $\vec{r}_v = \frac{\partial x}{\partial v}(u,v)\vec{i} + \frac{\partial y}{\partial v}(u,v)\vec{j} + \frac{\partial z}{\partial v}(u,v)\vec{k}$. If we fix $v = v_0$ then $\vec{r}_u(u,v_0)$ is tangent to the curve $\vec{r}(u,v_0)$ provided that $\vec{r}_u(u,v_0) \neq \vec{0}$. Similarly if $u = u_0$ then $\vec{r}_v = (u_0,v)$ is tangent to the curve $\vec{r}(u_0,v)$ provided that $\vec{r}_v(u_0,v) \neq \vec{0}$. Therefore, both $\vec{r}_u(u_0,v_0)$ and $\vec{r}_v(u_0,v_0)$ are both tangent to the surface given neither are $\vec{0}$. This means that $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ and the vector $\vec{r}_u \times \vec{r}_v$ will be orthogonal to the surface so it can be the normal vector used for the tangent plane.

example. Find the equation of the tangent plane to the surface given by $\vec{r}(u,v) = u\vec{i} + 2v^2\vec{j} + (u^2+v)\vec{k}$ at the point $(2,2,3)$.

First, compute the point(s) (u,v) that gives $(2,2,3)$:
$2 = u \Rightarrow u = 2$
$2 = 2v^2 \Rightarrow v = \pm 1$
$3 = u^2 + v$
$3 = 4 + v \Rightarrow v = -1$

Second, compute $\vec{r}_u$, $\vec{r}_v$, and $\vec{n} = \vec{r}_u \times \vec{r}_v$:
$\vec{r}_u(u,v) = \vec{i} + 2u\vec{k}$
$\vec{r}_v(u,v) = 4v\vec{j} + \vec{k}$
$\vec{n} = \vec{r}_u \times \vec{r}_v = -8uv\vec{i} - \vec{j} + 4v\vec{k}$

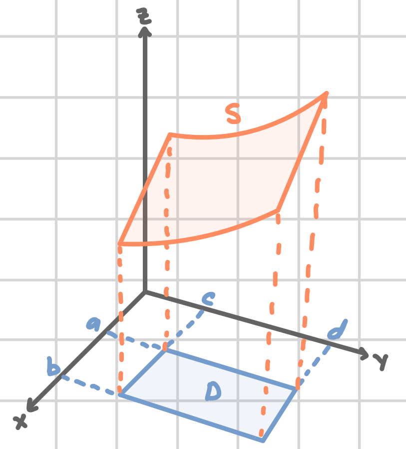
Write out tangent plane:
$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$ OR $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$
$\vec{n}(2,-1) = 16\vec{i} - \vec{j} - 4\vec{k}$
$16(x-2) - (y-2) - 4(z-3) = 0$

We can also use the equations to calculate the surface area of the parametric surface:
Provided S is traced out exactly once as (u,v) ranges over the points in D, the surface area of S is given by $A = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA$.

example. Find the surface area of the portion of the sphere of radius 4 that lies inside the cylinder $x^2+y^2 = 12$ and above the xy-plane.
The parameterization of the sphere is given by $\vec{r}(\theta,\varphi) = 4\sin\varphi\cos\theta\,\vec{i} + 4\sin\varphi\sin\theta\,\vec{j} + 4\cos\varphi\,\vec{k}$.
Determine D: we are not restricting how far around the z-axis we are rotating so we take $0 \leq \theta \leq 2\pi$.
To find the range for φ, we must find the intersection of the sphere and the cylinder:
$x^2+y^2+z^2 = 16 \Rightarrow 12 + z^2 = 16 \Rightarrow z^2 = 4 \Rightarrow z = \pm 2$ but z is above the xy-plane so $z = 2$.
Using $z = \rho\cos\varphi$, $z = 2$ and $\rho = 4$, we have $2 = 4\cos\varphi \Rightarrow \varphi = \frac{\pi}{3}$. Therefore $0 \leq \varphi \leq \pi/3$ is the range.
Now we need to find $\vec{r}_\theta \times \vec{r}_\varphi$: $\vec{r}_\theta(\theta,\varphi) = -4\sin\varphi\sin\theta\,\vec{i} + 4\sin\varphi\cos\theta\,\vec{j}$ and $\vec{r}_\varphi(\theta,\varphi) = 4\cos\varphi\cos\theta\,\vec{i} + 4\cos\varphi\sin\theta\,\vec{j} - 4\sin\varphi\,\vec{k}$.
$\vec{r}_\theta \times \vec{r}_\varphi = -16\sin^2\varphi\cos\theta\,\vec{i} - 16\sin\varphi\cos\varphi\sin^2\theta\,\vec{k} - 16\sin^2\varphi\sin\theta\,\vec{j} - 16\sin\varphi\cos\varphi\cos^2\theta\,\vec{k} = -16\sin^2\varphi\cos\theta\,\vec{i} - 16\sin^2\varphi\sin\theta\,\vec{j} - 16\sin\varphi\cos\varphi\,\vec{k}$.
$\|\vec{r}_\theta \times \vec{r}_\varphi\| = \sqrt{256\sin^4\varphi\cos^2\theta + 256\sin^4\varphi\sin^2\theta + 256\sin^2\varphi\cos^2\varphi} = \sqrt{256\sin^4\varphi(\cos^2\theta+\sin^2\theta) + 256\sin^2\varphi\cos^2\varphi} = \sqrt{256\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)}$
$= 16\sqrt{\sin^2\varphi} = 16|\sin\varphi| = 16\sin\varphi$.
Compute the surface area: $A = \iint_D 16\sin\varphi \, dA = \int_0^{2\pi}\int_0^{\pi/3} 16\sin\varphi \, d\varphi \, d\theta$
$= \int_0^{2\pi} \left[-16\cos\varphi\right]_0^{\pi/3} d\theta$
$= \int_0^{2\pi} 8 \, d\theta$
$= 16\pi$

Surface Integrals

We now shift our focus to integrating over some surface S in three-dimensional space. I am providing a sketch of a surface that lies above some region D in the xy-plane that resembles a rectangle, but D does not have to be a rectangle and we can view the surface as being in front of some region D in the yz-plane or the xz-plane.



There are two methods used to evaluate the surface integral depending how it is given.

First, given the surface integral where S is given by $z=g(x,y)$ we have the formula:

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}\,dA.$$

This is utilizing the easy parameterization given in the box above, similar formulas exist for the others.

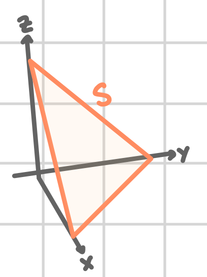
The second version utilizes the parameterization $\vec{r}(u,v)=x(u,v)\vec{i}+y(u,v)\vec{j}+z(u,v)\vec{k}$ and region D.

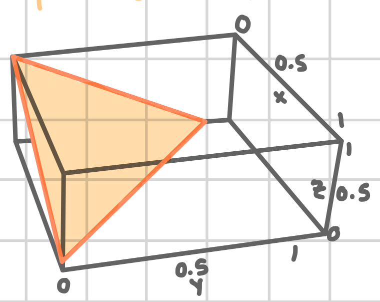
$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\cdot\|\vec{r}_u\times\vec{r}_v\|\,dA.$$

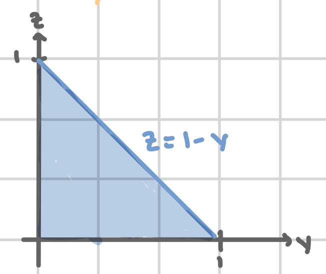
These are really the same thing since $\|\vec{r}_x\times\vec{r}_y\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}$ when $\vec{r}(x,y)=x\vec{i}+y\vec{j}+g(x,y)\vec{k}$ is the given parameterization.

example. Evaluate $\iint_S 6xy\,dS$ where S is the portion of the plane $x+y+z=1$ that lies in the 1^{st} octant and is in front of the yz-plane.

We want the portion of the plane that lies in front of the yz-plane so we rewrite the equation in the form $x=f(y,z)$, $x=1-y-z$. To help us determine the region D, we sketch the surface S and the region D in the yz-plane.







We get the region D by setting $z=0$.

$0\le y\le 1$

$0\le z\le 1-y$

This is one of those similar formula times: $\iint_S f(x,y,z)\,dS = \iint_D f(g(y,z),y,z)\sqrt{\left(\frac{\partial g}{\partial y}\right)^2+\left(\frac{\partial g}{\partial z}\right)^2+1}\,dA.$

$$\begin{aligned}
\iint_S 6xy\,dS &= \iint_D 6(1-y-z)y\sqrt{1+(-1)^2+(-1)^2}\,dA\\
&= 6\sqrt{3}\int_0^1\int_0^{1-y} y-y^2-yz\;dz\,dy\\
&= 6\sqrt{3}\int_0^1 yz-y^2z-\tfrac{1}{2}yz^2\Big]_0^{1-y}\,dy\\
&= 6\sqrt{3}\int_0^1 y(1-y)-y^2(1-y)-\tfrac{1}{2}(1-y)^2\,dy\\
&= 6\sqrt{3}\int_0^1 \tfrac{1}{2}y-y^2+\tfrac{1}{2}y^3\,dy\\
&= 6\sqrt{3}\left[\tfrac{1}{4}y^2-\tfrac{1}{3}y^3+\tfrac{1}{8}y^4\right]_0^1\\
&= 6\sqrt{3}\left(\tfrac{1}{4}-\tfrac{1}{3}+\tfrac{1}{8}\right)\\
&= \frac{\sqrt{3}}{4}
\end{aligned}$$

example. Evaluate $\iint_S z\,dS$ where S is the upper half of a sphere of radius 2.

Using the parameterization of the sphere from above: $\vec{r}(\theta,\varphi)=2\sin\varphi\cos\theta\,\vec{i}+2\sin\varphi\sin\theta\,\vec{j}+2\cos\varphi\,\vec{k}$ with parameter ranges $0\le\theta\le 2\pi$ and $0\le\varphi\le\pi/2$.

We need to find $\|\vec{r}_\theta\times\vec{r}_\varphi\|$:

$\vec{r}_\theta(\theta,\varphi)=-2\sin\varphi\sin\theta\,\vec{i}+2\sin\varphi\cos\theta\,\vec{j}$

$\vec{r}_\varphi(\theta,\varphi)=2\cos\varphi\cos\theta\,\vec{i}+2\cos\varphi\sin\theta\,\vec{j}-2\sin\varphi\,\vec{k}$

$\vec{r}_\theta\times\vec{r}_\varphi=-4\sin^2\varphi\cos\theta\,\vec{i}-4\sin\varphi\cos\varphi\sin^2\theta\,\vec{k}-4\sin^2\varphi\sin\theta\,\vec{j}-4\sin\varphi\cos\varphi\cos^2\theta\,\vec{k}=-4\sin^2\varphi\cos\theta\,\vec{i}-4\sin^2\varphi\sin\theta\,\vec{j}-4\sin\varphi\cos\varphi\,\vec{k}$

$\|\vec{r}_\theta\times\vec{r}_\varphi\|=\sqrt{16\sin^4\varphi\cos^2\theta+16\sin^4\varphi\sin^2\theta+16\sin^2\varphi\cos^2\varphi}=4\sin\varphi$

Compute
$$\begin{aligned}
\iint_S z\,dS &= \iint_D 2\cos\varphi\,(4\sin\varphi)\,dA\\
&= \int_0^{2\pi}\int_0^{\pi/2} 4\sin(2\varphi)\,d\varphi\,d\theta\\
&= \int_0^{2\pi}\left[-2\cos(2\varphi)\right]_0^{\pi/2}d\theta\\
&= \int_0^{2} 4\,d\theta\\
&= \left[4\theta\right]_0^2\\
&= 8
\end{aligned}$$

Surface Integrals of Vector Fields

Just like with line integrals, we want to do surface integrals of vector fields. Recall that orientation was important during line integrals. The same will be true here.

def. A surface S is closed if it is the boundary of some solid region E.

def. We say that the closed surface S has a positive orientation if we can choose a set of unit normal vectors that point outward from the region E while the negative orientation will be the set of unit normal vectors that point in towards the region E.

Unit Normal Vector

Way 1: Suppose we are given a surface defined by $z=g(x,y)$ We can write this as $f(x,y,z)=z-g(x,y)$. Then the surface can be described by $f(x,y,z)=0$. Recall that ∇f is orthogonal to a surface given by $f(x,y,z)=0$, so we have a normal vector to the surface. But this may not be a unit normal vector, thus we take $\vec{n}=\frac{\nabla f}{\|\nabla f\|}=\frac{-g_x\vec{i}-g_y\vec{j}+\vec{k}}{\sqrt{(g_x)^2+(g_y)^2+1}}$.

If given $y=g(x,z)$, use $f(x,y,z)=y-g(x,z)$. If given $x=g(y,z)$, use $f(x,y,z)=x-g(y,z)$.

Way 2: Suppose we are given the parametric representation $\vec{r}(u,v)=x(u,v)\vec{i}+y(u,v)\vec{j}+z(u,v)\vec{k}$. We use the fact that the vector $\vec{r}_u\times\vec{r}_v$ is normal to the tangent plane at a particular point. This counts as a normal vector, but we can not guarantee that it is a unit normal vector so take $\vec{n}=\frac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}$.

With both cases we must make sure the vector points in the correct direction (take the negative if not).

Given a vector field $\vec{F}$ with unit normal vector $\vec{n}$ then the surface integral of $\vec{F}$ over the surface S is given by,

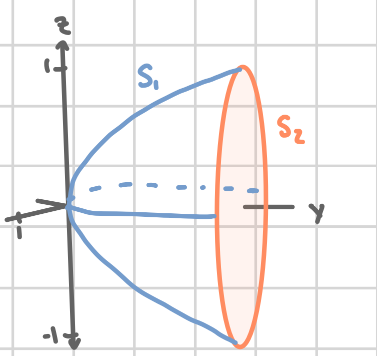
$\iint_S \vec{F}\cdot d\vec{S}=\iint_S \vec{F}\cdot\vec{n}\,dS$ where dS denotes the standard surface integral. This is commonly called the flux of $\vec{F}$ across S.

Way 1 gives a more simplified formula: $\iint_S \vec{F}\cdot d\vec{S}=\iint_S \vec{F}\cdot\vec{n}\,dS=\iint_D (P\vec{i}+Q\vec{j}+R\vec{k})\cdot\left(\frac{-g_x\vec{i}-g_y\vec{j}+\vec{k}}{\sqrt{(g_x)^2+(g_y)^2+1}}\right)\sqrt{(g_x)^2+(g_y)^2+1}\,dA$

$=\iint_D (P\vec{i}+Q\vec{j}+R\vec{k})\cdot(-g_x\vec{i}-g_y\vec{j}+\vec{k})\,dA=\iint_D -Pg_x-Qg_y+R\,dA$

Way 2 also simplifies to another formula: $\iint_S \vec{F}\cdot d\vec{S}=\iint_S \vec{F}\cdot\vec{n}\,dS=\iint_D \vec{F}\cdot\left(\frac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}\right)\|\vec{r}_u\times\vec{r}_v\|\,dA=\iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$

example. Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}=y\vec{j}-z\vec{k}$ and S is the surface given by the paraboloid $y=x^2+z^2$, $0\le y\le 1$ and the disk $x^2+z^2\le 1$ at $y=1$. Assume that S has positive orientation.



The disk $x^2+z^2\le 1$ caps off the paraboloid $y=x^2+z^2$ which makes this a closed surface.

We need to split the integral into the surface integral on each surface and add them together.

S_1: Use $f(x,y,z)=y-g(x,z)=y-x^2-z^2$. Then $\nabla f=\langle -2x,1,-2z\rangle$ and we can conclude $\vec{n}=\frac{-\nabla f}{\|-\nabla f\|}=\frac{\langle 2x,-1,2z\rangle}{\|\nabla f\|}$.

$\iint_{S_1}\vec{F}\cdot d\vec{S}=\iint_D (y\vec{j}-z\vec{k})\cdot\left(\frac{\langle 2x,-1,2z\rangle}{\|\nabla f\|}\right)\|\nabla f\|\,dA=\iint_D -y-2z^2\,dA=\iint_D -(x^2+z^2)-2z^2\,dA=-\iint_D x^2+3z^2\,dA$

$=-\int_0^{2\pi}\int_0^1 (r^2\cos^2\theta+3r^2\sin^2\theta)\,r\,dr\,d\theta=-\int_0^{2\pi}\left(\frac{1}{2}(1+\cos(2\theta))+\frac{3}{2}(1-\cos(2\theta))\right)\left(\frac{1}{4}r^4\right)\Big]_0^1 d\theta$

$=-\frac{1}{8}\int_0^{2\pi}4-2\cos(2\theta)\,d\theta=-\frac{1}{8}(4\theta-\sin(2\theta))\Big]_0^{2\pi}=-\pi.$

S_1: $x=r\cos\theta$ $\quad z=r\sin\theta$

$0\le\theta\le 2\pi$ $\quad 0\le r\le 1$

S_2: $0\le\theta\le 2\pi$ $\quad 0\le r\le 1$

S_2: The disk is just the portion of the $y=1$ plane that is in front of the disk of radius 1 in the xz-plane. We want the unit normal vector to point away from the enclosed surface and parallel to the y-axis. Thus we can take $\vec{n}=\vec{j}$.

$\iint_{S_2}\vec{F}\cdot d\vec{S}=\iint_{S_2}(y\vec{j}-z\vec{k})\cdot(\vec{j})\,dS=\iint_{S_2} y\,dS=\iint_D 1\cdot\sqrt{0+1+0}\,dA=\iint_D dA=\pi$

$S=S_1\cup S_2$: $\iint_S \vec{F}\cdot d\vec{S}=\iint_{S_1}\vec{F}\cdot d\vec{S}+\iint_{S_2}\vec{F}\cdot d\vec{S}=-\pi+\pi=0.$