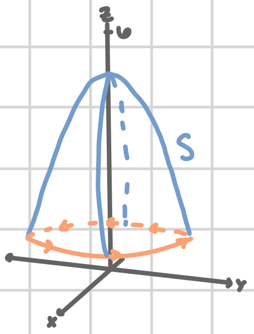
Standard 21: Stoke's Theorem

Stoke's Theorem

This section covers a higher dimensional version of Green's theorem. We are going to relate a line integral to a surface integral. The curve that we utilize in the line integral is called the boundary curve. The orientation of the surface S will induce the positive orientation of C. The positive orientation of C is thought of as walking along the curve with your head pointed in the same directon as the unit normal vectors while the surface on the left.

Stoke's Theorem. Let S be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Also let $\vec{F}$ be a vector field then, $\int_C \vec{F} \cdot dr = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$.

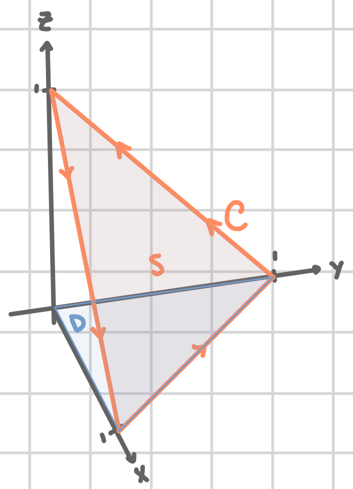
example. Use Stokes' Theorem to evaluate $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ where $\vec{F} = z^2\vec{i} - 3xy\vec{j} + x^3y^3\vec{k}$ and S is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that S is oriented upwards.



The boundary curve C will be where the surface intersects the plane $z=1$: $x^2+y^2=4$ at $z=1$ with parameterization $\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + \vec{k}$ for $0 \le t \le 2\pi$.

By Stokes' Theorem:
$$\begin{aligned}\iint_S \text{curl}\vec{F} \cdot d\vec{S} &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt \\ &= \int_0^{2\pi} (\vec{i} - 12\cos t\sin t\,\vec{j} + 64\cos^3 t\sin^3 t\,\vec{k}) \cdot (-2\sin t\,\vec{i} + 2\cos t\,\vec{j})\,dt \\ &= \int_0^{2\pi} -2\sin t - 24\sin t\cos^2 t\,dt \\ &= \left[2\cos t + 8\cos^3 t\right]_0^{2\pi} \\ &= 0\end{aligned}$$

example. Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = z^2\vec{i} + y^2\vec{j} + x\vec{k}$ and C is the triangle with vertices (1,0,0), (0,1,0), and (0,0,1) with counter-clockwise rotation.



The equation of the plane is $x+y+z=1$ which can be written $z = g(x,y) = 1-x-y$.

$$\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & x \end{vmatrix} = 2z\vec{j} - \vec{j} = (2z-1)\vec{j}$$

$f(x,y,z) = z - g(x,y) = z - 1 + x + y$

$\nabla f = \vec{i} + \vec{j} + \vec{k}$

D: $0 \le x \le 1$ $\quad 0 \le y \le -x+1$

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \iint_S \text{curl}\vec{F} \cdot d\vec{S} \\ &= \iint_S (2z-1)\vec{j} \cdot d\vec{S} \\ &= \iint_S (2z-1) \cdot \frac{\vec{i}+\vec{j}+\vec{k}}{\|\nabla f\|}\|\nabla f\|\,dA \\ &= \int_0^1\int_0^{-x+1} 2(1-x-y) - 1\,dy\,dx \\ &= \int_0^1 \left[y - 2xy - y^2\right]_0^{-x+1} dx \\ &= \int_0^1 x^2 - x\,dx \\ &= \left(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right)\Big]_0^1 \\ &= -\tfrac{1}{6}\end{aligned}$$