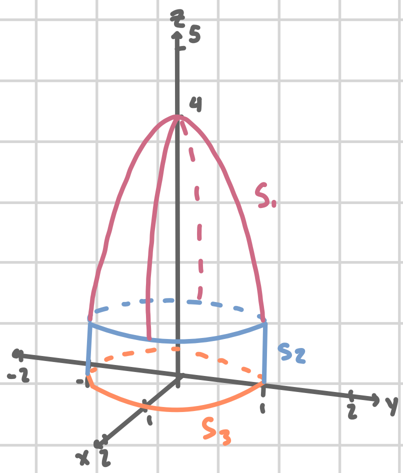
Standard 22: Divergence Theorem

Divergence Theorem

Divergence Theorem Let E be a simple solid region and S is the boundary surface of E with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous first order partial derivatives. Then, $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$.

example. Use the divergence theorem to evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = xy\vec{i} - \frac{1}{2}y^2\vec{j} + z\vec{k}$ and the surface consists of the three surfaces, $z = 4 - 3x^2 - 3y^2$, $1 \le z \le 4$ on the top, $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides and $z = 0$ on the bottom.



In cylindrical coordinates:

$0 \le z \le 4 - 3r^2$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$\operatorname{div}\vec{F} = y - y + 1 = 1$

$$
\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \iiint_E \operatorname{div}\vec{F}\, dV \\
&= \int_0^{2\pi}\int_0^1\int_0^{4-3r^2} r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 rz\Big]_0^{4-3r^2} dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 4r - 3r^3\, dr\, d\theta \\
&= \int_0^{2\pi} 2r^2 - \frac{3}{4}r^4\Big]_0^1 d\theta \\
&= \int_0^{2\pi} \frac{5}{4}\, d\theta \\
&= \frac{5}{4}\theta\Big]_0^{2\pi} \\
&= \frac{5}{2}\pi
\end{aligned}
$$