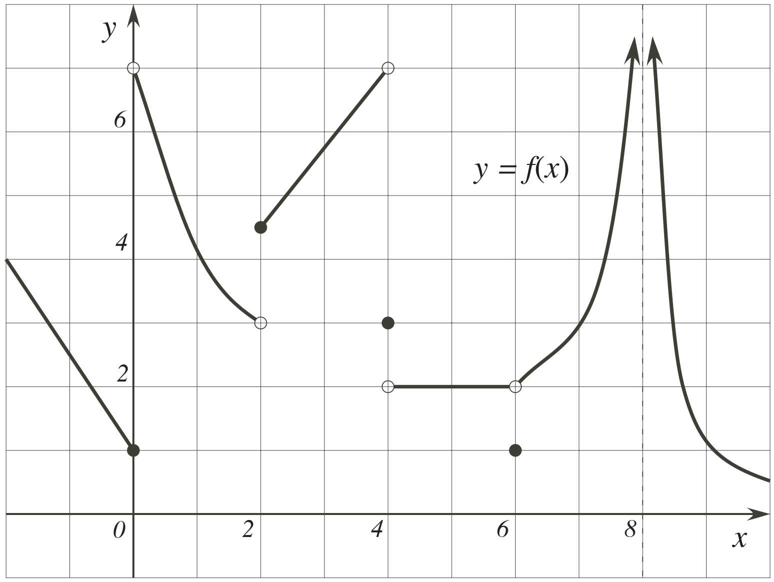
10350 In-Class Exam 01 Review



1. The graph of $y = f(x)$ for $-2 \leq x \leq 10$ is shown above. Find the values of the following limits if it exists:

(a) $\lim_{x \to 6} [2f(x)] \stackrel{?}{=}$ $2[\lim_{x\to 6} f(x)] = 2[2] = 4$

(b) $\lim_{x \to 2^-} [x \cdot f(x)] \stackrel{?}{=}$ $\lim_{x\to 2^-} x \cdot \lim_{x\to 2^-} f(x) = 2 \cdot 3 = 6$

(c) $\lim_{x \to 3} \dfrac{f(x) - f(3)}{x - 3} \stackrel{?}{=}$ $\dfrac{\text{rise}}{\text{run}} = \dfrac{2.5}{2} = \dfrac{\frac{5}{2}}{2} = \dfrac{5}{4}$

(d) $\lim_{x \to 0^+} \dfrac{\sqrt{f(x)}}{f(x) + 1} \stackrel{?}{=}$ $\dfrac{\lim_{x\to 0^+} \sqrt{f(x)}}{\lim_{x\to 0^+} f(x)+1} = \dfrac{\sqrt{7}}{7+1} = \dfrac{\sqrt{7}}{8}$

$\lim_{x\to 0^+} f(x) = 7$

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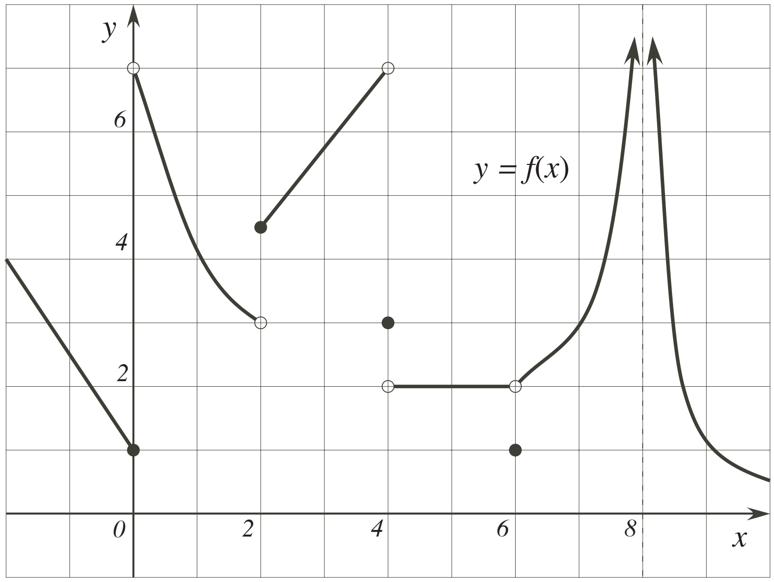

2. The graph of $y = f(x)$ for $-2 < x < 10$ is shown above.

a. Find all values of x in $(-2, 10)$ for which the function $y = f(x)$ is discontinuous.

$x = 0, 2, 4, 6, 8$

b. Amongst the values of x in (a), which of them are places where there is a **removable discontinuity**?

$\lim_{x \to c} f(x) \neq f(c)$

$x = 6$

c. Amongst the values of x in (a), which of them are places where there is a **jump discontinuity**?

$\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$

$x = 0, 2, 4$

$x=8$ is a special case:
$\lim_{x \to 8^-} f(x) = \infty$ and $\lim_{x \to 8^+} f(x) = \infty$ so $\lim_{x \to 8} f(x) = \infty$, but we can not "plug" this hole with $(8, \infty)$. In fact it is deemed as neither a jump or a removable discontinuity.

d. Amongst the values of x in (a), which of them are places where $f(x)$ is **left** or **right** continuous? Explain your choices.

Left continuous:

$x = 0$

Right continuous:

$x = 2$

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3. The position of a particle at time t seconds moving on a straight line is given by $s(t) = \dfrac{1}{2t+1}$ meters. Answer the following questions below.

a. Find the average velocity of the particle between $t = 2$ and $t = 2 + h$. Simplify your answers as far as possible assuming $h \neq 0$.

average: $\dfrac{f(b) - f(a)}{b - a}$

$$\frac{f(2+h) - f(2)}{2+h-2}$$

$$= \frac{\frac{1}{2(2+h)+1} - \frac{1}{2(2)+1}}{h}$$

$$= \frac{\frac{1}{4+2h+1} - \frac{1}{5}}{h}$$

$$= \frac{\frac{1}{2h+5} - \frac{1}{5}}{h}$$

$$= \frac{\frac{5 - (2h+5)}{5(2h+5)}}{\frac{h}{1}}$$

$$= \frac{\frac{5-2h-5}{5(2h+5)}}{\frac{h}{1}}$$

$$= \frac{-2h}{5(2h+5)} \cdot \frac{1}{h}$$

$$= \frac{-2}{10h+25}$$

b. Find the difference quotient of $s(t)$ at $t = 2$.

difference quotient: $\dfrac{f(x+h) - f(x)}{h}$

$$= \frac{\frac{1}{2(t+h)+1} - \frac{1}{2t+1}}{h}$$

$$= \frac{\frac{1}{2t+2h+1} - \frac{1}{2t+1}}{h}$$

$$= \frac{\frac{2t+1-(2t+2h+1)}{(2t+2h+1)(2t+1)}}{h}$$

$$= \frac{\frac{\cancel{2t}\cancel{+1}-\cancel{2t}-2h\cancel{-1}}{(2t+2h+1)(2t+1)}}{\frac{h}{1}}$$

$$= \frac{-2\cancel{h}}{(2t+2h+1)(2t+1)} \cdot \frac{1}{\cancel{h}}$$

$$= \frac{-2}{(2t+2h+1)(2t+1)}$$

@ $t = 2$:

$$= \frac{-2}{(2(2)+2h+1)(2(2)+1)}$$

$$= \frac{-2}{(2h+5)(5)}$$

$$= \frac{-2}{10h+25}$$

c. Explain using limits how one can find the slope of the tangent line to $s(t) = \dfrac{1}{2t+1}$ at $t = 2$.

$\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$

$= \lim_{h \to 0} \dfrac{-2}{10h+25}$

$= \dfrac{-2}{25}$

d. Find the instantaneous velocity of the particle at $t = 2$. Draw a graph to illustrate your answer.

IROC: $\lim_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$

@ $t=2$: $\lim_{h \to 0} \dfrac{-2}{10h+25}$

$= \dfrac{-2}{25}$

e. Find the equation of the tangent line to $s(t) = \dfrac{1}{2t+1}$ at $t = 2$. .

tangent line: $y - f(x_1) = f'(x_1)(x - x_1)$

$m = -\dfrac{2}{25}$

$x_1 = 2$

$y_1 = \dfrac{1}{2(2)+1} = \dfrac{1}{5}$

$y - \dfrac{1}{5} = -\dfrac{2}{25}(x-2)$

$y - \dfrac{1}{5} = -\dfrac{2}{25}x - \dfrac{4}{25}$

$+\dfrac{1}{5} \qquad +\dfrac{5}{25}$

$y = -\dfrac{2}{25}x + \dfrac{1}{25}$

4. Solve for x at which the two curves intersect: $\begin{cases} y &= 3\ln(x) - 2 \\ y &= \ln(3 - x^3) \end{cases}$

$3\ln(x) - 2 = \ln(3 - x^3)$

$\ln(x^3) - \ln(3 - x^3) = 2$

$\ln\left(\frac{x^3}{3 - x^3}\right) = 2$

$\cancel{e^{\ln}}\left(\frac{x^3}{3 - x^3}\right) = e^2$

$\frac{x^3}{3 - x^3} = e^2$

$x^3 = e^2(3 - x^3)$

$x^3 = 3e^2 - e^2x^3$

$x^3 + e^2x^3 = 3e^2$

$x^3(1 + e^2) = 3e^2$

$x^3 = \frac{3e^2}{1 + e^2}$

$x = \sqrt[3]{\frac{3e^2}{1 + e^2}}$

5. Recall the compound interest formulae: $B = P\left(1+\dfrac{r}{n}\right)^{nt}$; $B = Pe^{rt}$.

Find the annual interest rate of an account that quadruples its principal in 10 years if interest is compounded (a) monthly and (b) continuously.

(a) quadruples at $t=10$: $4P = P(1+\frac{r}{12})^{12(10)}$

$4 = (1+\frac{r}{12})^{120}$

$4^{1/120} = (1+\frac{r}{12})$

$4^{1/120} - 1 = \frac{r}{12}$

$12(4^{1/120}-1) = r$

(b) quadruples at $t=10$: $4P = Pe^{r\cdot 10}$

$4 = e^{10r}$

$\ln(4) = \cancel{\ln(e}^{10r})$

$\ln(4) = 10r$

$\frac{\ln(4)}{10} = r$

6. Consider the function

$$f(x) = \begin{cases} a(2^x) + 6 & -\infty < x \leq 1 \\ \dfrac{2x^2 - x - 1}{x - 1} & 1 < x < +\infty \end{cases}$$

Find the value of a such that $f(x)$ is continuous at $x = 1$.

continuous: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

given: left continuity i.e. $\lim_{x \to 1^-} f(x) = f(1)$

find a such that: $\lim_{x \to 1^+} f(x) = f(1)$

$$\lim_{x \to 1^+} \frac{2x^2 - x - 1}{x - 1} = a(2^1) + 6$$

$$\lim_{x \to 1^+} \frac{(2x+1)(x-1)}{x-1} = 2a + 6$$

$$\lim_{x \to 1^+} (2x+1) = 2a + 6$$

$$2(1) + 1 = 2a + 6$$

$$3 = 2a + 6$$

$$-3 = 2a$$

$$-\frac{3}{2} = a$$

7. Find the equation of the tangent line to the graph of $y = 8e^x - 2x + 1$ which is parallel to the line whose equation is $-6x + y = 4$. Give your answer in the form $y = mx + b$.

parallel = same slope, tangent slope = derivative

find x where $f'(x)$ = slope of given line

Step 1: Find slope of given line

$-6x + y = 4$

$y = 6x + 4$

$m = 6$

Step 2: Find x_1 s.t. $f'(x_1) = m$

$f'(x) = 8e^x - 2 = 6$

$8e^x = 8$

$e^x = 1$

$\cancel{\ln(e^x)} = \ln(1)$

$x_1 = 0$

Step 3: Find $f(x_1)$

$f(0) = 8e^0 - 2(0) + 1$

$= 8(1) - 0 + 1$

$= 8 + 1$

$= 9$

Step 4: Plug into $y - f(x_1) = f'(x)(x - x_1)$

$y - 9 = 6(x - 0)$

$y - 9 = 6x$

$y = 6x + 9$

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8. The profit, in thousands of dollars, from the sales of a certain ipod accessory is given by the formula

$$P(x) = -2x^2 + 12x - 13$$

where x is the number of dozens accessory sold.

(a) By completing the square, write $P(x)$ in the form $P(x) = a(x+b)^2 + c$. Show clearly all your steps.

$\hookrightarrow (x + \frac{b}{2})^2 = x^2 + bx + (\frac{b}{2})^2$

$P(x) = -2x^2 + 12x - 13$

factor out a

$= -2(\underline{x^2 - 6x}) - 13$

complete the square

$b = -6$, $(\frac{1}{2}b)^2 = (-3)^2 = 9$

$= -2(x^2 - 6x + 9 - 9) - 13$

$= -2((x-3)^2 - 9) - 13$

$= -2(x-3)^2 + 18 - 13$

$= -2(x-3)^2 + 5$

(b) Find the number of dozens of accessory that must be sold so profit is at its maximum. What is the maximum profit?

maximum = vertex = $(-b, c)$

maximum = $(3, 5)$

the maximum profit is 5 thousand dollars when 3 dozen accessories are sold