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1a. Find $\dfrac{dy}{dx}$ if $\sec(y) + e^{xy} = y + \sqrt{2} - \dfrac{\pi}{4} + e^{\pi/4}$.

$$\sec(y)\cdot\tan(y)\frac{dy}{dx} + e^{xy}\left[(1)(y) + \left(\frac{dy}{dx}\right)(x)\right] = \frac{dy}{dx} + 0 + 0 + 0$$

$$\sec(y)\cdot\tan(y)\cdot\frac{dy}{dx} + ye^{xy} + x\frac{dy}{dx} = \frac{dy}{dx}$$

$$\sec(y)\cdot\tan(y)\cdot\frac{dy}{dx} + x\frac{dy}{dx} - \frac{dy}{dx} = -ye^{xy}$$

$$\frac{dy}{dx}\left(\sec(y)\cdot\tan(y) + x - 1\right) = -ye^{xy}$$

$$\frac{dy}{dx} = \frac{-ye^{xy}}{\sec(y)\tan(y) + x - 1}$$

1b. Find the equation of the tangent line to the curve $\sec(y) + e^{xy} = y + \sqrt{2} - \dfrac{\pi}{4} + e^{\pi/4}$ at the point $\left(1, \dfrac{\pi}{4}\right)$.

$$\left.\frac{dy}{dx}\right|_{(1,\pi/4)} = \frac{-\frac{\pi}{4}e^{(1)(\pi/4)}}{\sec(\pi/4)\tan(\pi/4) + 1 - 1} = -\frac{\pi}{4}\left(\frac{e^{\pi/4}}{\sec(\pi/4)\tan(\pi/4)}\right)$$

$$= -\frac{\pi}{4}\left(\frac{e^{\pi/4}}{(2/\sqrt{2})\cdot 1}\right)$$ multiply by reciprocal

$$= -\frac{\pi}{4}\left(\frac{e^{\pi/4}\sqrt{2}}{2}\right)$$

$$= \frac{-\pi e^{\pi/4}}{8}$$

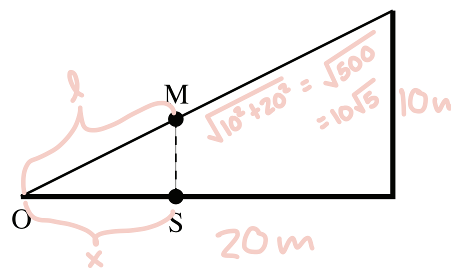
$$y - f(x_1) = f'(x_1)(x - x_1)$$

$$y - \frac{\pi}{4} = \frac{-\pi e^{\pi/4}}{8}(x - 1)$$

2. A rope is tied to the top of a 10 meter tall structure and the other end anchored to the ground O at a point 20 meters from the base of the structure. A monkey climbs along the rope casting a shadow on the ground directly below it. Find how fast the monkey is climbing along the rope when its shadow is 6 meters from O if its shadow~~ing~~ is moving at a rate of 1/4 meter/sec towards the base of a the 10 meter structure. Assume there is no slack in the rope and the structure is perpendicular to the ground.

Given $\frac{dx}{dt} = \frac{1}{4}$ m/s when $x=6$

Find $\frac{d\ell}{dx} = ?$ when $x=6$



Step1: Relate

ratio: $\frac{\ell}{\text{total}} = \frac{x}{\text{total}}$

$$\frac{\ell}{10\sqrt{5}} = \frac{x}{20}$$

$$\ell = 10\sqrt{5} \cdot \frac{x}{20}$$

$$= \frac{\sqrt{5}}{2} \cdot x$$

Step2: Derive

$$\frac{d\ell}{dt} = \frac{\sqrt{5}}{2} \cdot \frac{dx}{dt}$$

Step3: Knowns

$$\frac{d\ell}{dt} = \frac{\sqrt{5}}{2} \cdot \frac{1}{4}$$

$$= \frac{\sqrt{5}}{8}$$

3. A cone with fixed 9 cm height has a radius that grows at a rate of $\frac{1}{2}$ cm/min. If the initial length of the radius is 4 cm, find how fast the volume of the cone is growing at time $t = 3$ minutes.

Given $h = 9$cm always, $\frac{dr}{dt} = \frac{1}{2}$ cm/s

Find $\frac{dV}{dt} = ?$ when $t=3$

Step1: Relate

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2 (9)$$

$$= 3\pi r^2$$

Step2: Derive

$$\frac{dV}{dt} = 6\pi r \frac{dr}{dt}$$

?: find radius at $t=3$

$$r = 4 + \tfrac{1}{2}t$$

$$= 4 + \tfrac{1}{2}(3)$$

$$= \tfrac{8}{2} + \tfrac{3}{2}$$

$$= \tfrac{11}{2}$$

Step3: Knowns

$$\frac{dV}{dt} = 6\pi(\,?\,)\left(\tfrac{1}{2}\right)$$

$$= 6\pi\left(\tfrac{11}{2}\right)\left(\tfrac{1}{2}\right)$$

$$= 3\pi\left(\tfrac{11}{2}\right)$$

$$= \frac{33}{2}\pi$$

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4. Using linear approximation, find an estimate for the value of $\sqrt[3]{27.3}$.

linearization: $f(x) \approx f'(a)(x-a) + f(a)$

Let $f(x) = \sqrt[3]{x}$ $a = 27$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f'(a) = \frac{1}{3\sqrt[3]{(27)^2}}$

$= \frac{1}{3 \cdot 9}$

$= \frac{1}{27}$

$f(27) = \sqrt[3]{27}$

$= 3$

$f(x) \approx \frac{1}{27}(x-27) + 3$

$f(x) \approx \frac{1}{27}x - 1 + 3$

$= \frac{1}{27}x + 2$

$f(27.3) \approx \frac{27.3}{27} + 2$

5. Find the derivative of $y = (2 + \cos x)^x$.

$y = (2+\cos x)^x$

$\ln(y) = \ln((2+\cos x)^x)$

$\ln(y) = x \cdot \ln(2+\cos x)$

$\frac{1}{y} \cdot \frac{dy}{dx} = (1)(\ln(2+\cos x)) + \frac{1}{2+\cos x} \cdot (-\sin x) \cdot (x)$

$\frac{dy}{dx} = (2+\cos x)^x \cdot \left[\ln(2+\cos x) - \frac{x\sin x}{2+\cos x}\right]$

6. A particle P is moving on the curve given by

$$2x^2y + 3y^3 = x^4 - 6.$$

Find how fast the particle is moving vertically at the point $(1, -1)$ when its horizontal velocity at $(1, -1)$ is 4 units/sec. Is the particle heading upward or downward at the location $(1, -1)$?

Given $\frac{dx}{dt} = 4$ units/sec at $(1,-1)$; find $\frac{dy}{dx} = ?$ at $(1,-1)$, is it positive or negative?

$2x^2y + 3y^3 = x^4 - 6$

$[(4x\frac{dx}{dt})(y) + (\frac{dy}{dt})(2x^2)] + 9y^2 \cdot \frac{dy}{dt} = 4x^3 \cdot \frac{dx}{dt} + 0$

$4(1)(4)(-1) + \frac{dy}{dt} \cdot 2(1)^2 + 9(-1)^2 \cdot \frac{dy}{dt} = 4(1)^3 \cdot (4)$

$-16 + 2\frac{dy}{dt} + 9\frac{dy}{dt} = 16$

$11 \cdot \frac{dy}{dt} = 32$

$\frac{dy}{dt} = \frac{32}{11}$

moving up the wall at $\frac{32}{11}$ unit/sec

7. Find the equation of the tangent line at $t = e$ for the curve given by the parametric equations:

$$x = 1 + \ln(t^3); \qquad y = \frac{2e}{t} = 2et^{-1}$$

Find also the cartesian equation of the curve given by the parametric equations.

eliminate t

(a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\frac{dx}{dt} = 0 + \frac{1}{t^3} \cdot 3t^2 = \frac{3}{t}$

$\frac{dy}{dt} = \frac{-2e}{t^2}$

$\left.\frac{dy}{dx}\right|_{t=e} = \frac{-2/e}{3/e} = \frac{-2}{\cancel{e}} \cdot \frac{\cancel{e}}{3} = \frac{-2}{3}$

$x(e) = 1 + \ln(e^3) = 1 + 3\ln(e) = 1 + 3 = 4$

at $t=e$: $\frac{dx}{dt} = \frac{3}{e}$

at $t=e$: $\frac{dy}{dt} = \frac{-2e}{e^2} = \frac{-2}{e}$

$y(e) = \frac{2e}{e} = 2$

$y - 2 = -\frac{2}{3}(x - 4)$

(b) eliminate t

$x = 1 + \ln(t^3)$

$y = \frac{2e}{t} \Rightarrow t = \frac{2e}{y}$

$x = 1 + \ln((\frac{2e}{y})^3)$

$x = 1 + 3\ln(\frac{2e}{y})$

$x = 1 + 3[\ln(2e) - \ln(y)]$

$x = 1 + 3\ln(2e) - 3\ln(y)$

$3\ln(y) = 1 + 3\ln(2e) - x$

$\ln(y^3) = 1 + 3\ln(2e) - x$

$y^3 = e^{1 + 3\ln(2e) - x}$

$y = (e^{1+3\ln(2e) - x})^{1/3}$

$y = e^{\frac{1}{3} + \ln(2e) - \frac{1}{3}x}$

$y = e^{\ln(2e)} \cdot e^{\frac{1}{3} - \frac{1}{3}x}$

$y = 2e^1 \cdot e^{\frac{1}{3} - \frac{1}{3}x}$

$y = 2e^{\frac{4}{3} - \frac{1}{3}x}$

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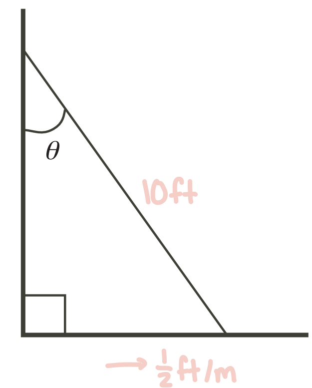
8. A 10 feet ladder leaning against a vertical wall at an angle θ is sliding in such a way that the other end on the ground is moving away from the base of the wall at 0.5 ft/min.

(a) How fast is the angle changing when the end on the ground is $5\sqrt{2}$ ft from the wall?

Given $\frac{dx}{dt} = \frac{1}{2}$ ft/min, $h = 10$ft always,

Find $\frac{d\theta}{dt} = ?$ when $x = 5\sqrt{2}$



1. Relation

$\sin\theta = \frac{x}{10}$

2. Implicitly derive

$\cos\theta \cdot \frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$ ⟶ find $\cos\theta$ when $x = 5\sqrt{2}$

$x^2 + y^2 = h^2$

$(5\sqrt{2})^2 + y^2 = 10^2$

$(25 \cdot 2) + y^2 = 100$

$50 + y^2 = 100$

$y^2 = 50$

$y = \pm\sqrt{50}$ negative distance does not make sense here

3. Knowns

$\frac{5\sqrt{2}}{10} \leftarrow \left(\frac{\sqrt{50}}{10}\right) \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot \frac{1}{2}$

$\frac{d\theta}{dt} = \frac{1}{20} \cdot \left(\frac{2}{\sqrt{2}}\right)$

$\frac{d\theta}{dt} = \frac{1}{10\sqrt{2}} = \frac{\sqrt{2}}{20}$

simplify $\sqrt{50}$

50 → 5, 10

10 → 5, 2

(b) How fast is the end on the wall moving when the end on the ground is $5\sqrt{2}$ ft from the wall?

Find $\frac{dy}{dt} = ?$ when $x = 5\sqrt{2}$

you can utilize the answer above and the relation $\cos\theta = \frac{y}{10}$, but any mistakes in (a) will make (b) harder

1. Relation

$x^2 + y^2 = 10^2$

2. Implicitly Derive

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

3. Knowns

$\cancel{2}(5\sqrt{2}) \cdot \frac{1}{\cancel{2}} + 2(5\sqrt{2}) \cdot \frac{dy}{dt} = 0$

$5\sqrt{2} + 10\sqrt{2} \cdot \frac{dy}{dt} = 0$

$10\sqrt{2} \cdot \frac{dy}{dt} = -5\sqrt{2}$

$\frac{dy}{dt} = -\frac{1}{2}$ ft/min

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9. Using the fact $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, discuss the continuity of the following function at $x = 0$:

$$f(x) = \begin{cases} \frac{\sin(3x)}{6x} & -\infty < x < 0 \\ \frac{1}{2} & x = 0 \\ \frac{\sin(4x)}{\sin(2x)} & 0 < x < \frac{\pi}{2} \end{cases}$$

$x=0$

$\lim_{x \to 0^-} \frac{\sin(3x)}{6x} = \lim_{x \to 0^+} \frac{\sin(4x)}{\sin(2x)}$

$\frac{3}{6} = \lim_{x \to 0^+} \frac{\sin(4x)}{x} \cdot \frac{x}{\sin(2x)}$

$\frac{1}{2} = \frac{4}{1} \cdot \frac{1}{2}$

$\frac{1}{2} = 2$

not continuous

jump discontinuity

$\lim_{x \to 0^-} \frac{\sin(3x)}{6x} = f(0)$

$\frac{1}{2} = \frac{1}{2}$

left continuous

$\lim_{x \to 0^+} \frac{\sin(4x)}{\sin(2x)} = f(0)$

$\frac{4}{2} = \frac{1}{2}$

$2 = \frac{1}{2}$

not right continuous

10.

x	-2.0	-1.5	-1.0	-0.5
$f(x)$	3.0	5.0	2.0	6.0

Selected values of a smooth function is given in the table above. Give as many estimate as you can for the slope of the graph of $f(x)$ at $x = -2, -1$, and -0.5.

$x=-2$

forward: $\frac{f(-1.5)-f(-2)}{-1.5-(-2)} = \frac{5-3}{0.5} = 2 \cdot 2 = 4$

$x=-1$

forward: $\frac{f(-0.5)-f(-1)}{-0.5-(-1)} = \frac{6-2}{0.5} = 4 \cdot 2 = 8$

backward: $\frac{f(-1)-f(-1.5)}{-1-(-1.5)} = \frac{2-5}{0.5} = -3 \cdot 2 = -6$

central: $\frac{f(-0.5)-f(-1.5)}{-0.5-(-1.5)} = \frac{6-5}{1} = 1$

$x=-0.5$

backward: $\frac{f(-0.5)-f(-1)}{-0.5-(-1)} = \frac{6-2}{0.5} = 4 \cdot 2 = 8$