10350 Exam 03 Review Set01 Name ______________

1. The statement: "$f'(x)$ is positive on $a < x < b$." then

1a. "$f(x)$ is increasing on $a < x < b$."

2. The statement: "$f'(x)$ is negative on $a < x < b$." then

2a. "$f(x)$ is decreasing on $a < x < b$."

2b. "The slope of the graph of $f(x)$ is negative on $a < x < b$."

3. The statement: "The graph of $f(x)$ is concave up on $a < x < b$." is the same as:

3a. "$f''(x)$ is positive on $a < x < b$." is the same as:

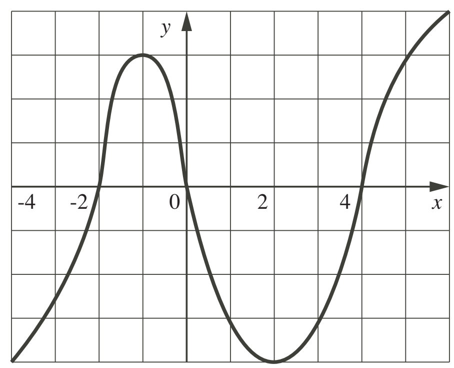
3b. "$f'(x)$ is increasing on $a < x < b$."

4. The statement: "$f'(x)$ is decreasing on $a < x < b$." is the same as:

4a. "$f''(x)$ is negative on $a < x < b$." is the same as:

4b. "The graph of $f(x)$ is concave down on $a < x < b$."

5. The figure below is the graph of the **derivative** $f'(x)$ of $f(x)$ for $-4 < x < 6$. Find all intervals on which **the graph of** $f(x)$ is concave up?



(i) Find all values of x in $(-4, 6)$ for which $f(x)$ is increasing.

$f'(x) > 0$

$(-2,0) \cup (4,6)$

(ii) Find the critical points of $f(x)$ in $(-4, 6)$. Are these local maximums or minimums?

$x = -2$ neg → pos local minimum

$x = 0$ pos → neg local maximum

$x = 4$ neg → pos local minimum

(local) minimums occur when $f'(x)$ goes from negative to positive.

maximums occur when $f'(x)$ goes from positive to negative.

(iii) Find all intervals on which **the graph of** $f(x)$ is concave up in $(-4, 6)$.

$f'(x)$ is increasing

$(-4,-1) \cup (2,6)$

(iv) Find all values of x in $(-4, 6)$ for which $f(x)$ has an inflection point.

$f''(x)$ changes signs

$x = -1, 2$

6. Find all critical points of $f(x) = 3\sqrt[3]{x^2 - 4}$. Classify all of them using first derivative test.

$f(x) = 3(x^2-4)^{1/3}$

$f'(x) = (x^2-4)^{-2/3} \cdot (2x)$

$0 = \dfrac{2x}{\sqrt[3]{(x^2-4)^2}}$ DNE when $x = \pm 2$

always positive

$0 = 2x$

$0 = x$

(local) minimum at $x = 0$

7. Find the maximum and minimum value of $f(x) = 3\sqrt[3]{x^2 - 4}$ for $-1 \le x \le 3$.

$f(-2) = 3\sqrt[3]{(-2)^2-4} = 3\sqrt{0} = 0$

$f(0) = 3\sqrt[3]{(0)^2-4} = 3\sqrt[3]{-4} = -3\sqrt[3]{4}$

$f(2) = 3\sqrt[3]{(2)^2-4} = 3\sqrt{0} = 0$

$f(-1) = 3\sqrt{(-1)^2-4} = 3\sqrt[3]{-3}$

$f(3) = 3\sqrt[3]{(3)^2-4} = 3\sqrt[3]{5}$

8. Draw a graph of a **continuous** function $y = f(x)$ except at $x = 2$ with the following properties. Indicate clearly its concavity and where it is increasing/decreasing.

a. The points $(0, 0)$ and $(4, 1)$ are on the graph.

b. $f'(0)$ does not exist. $f'(4) = 0$.

c. $\lim_{x\to 2^-} f(x) = -\infty$ and $\lim_{x\to 2^+} f(x) = +\infty$.

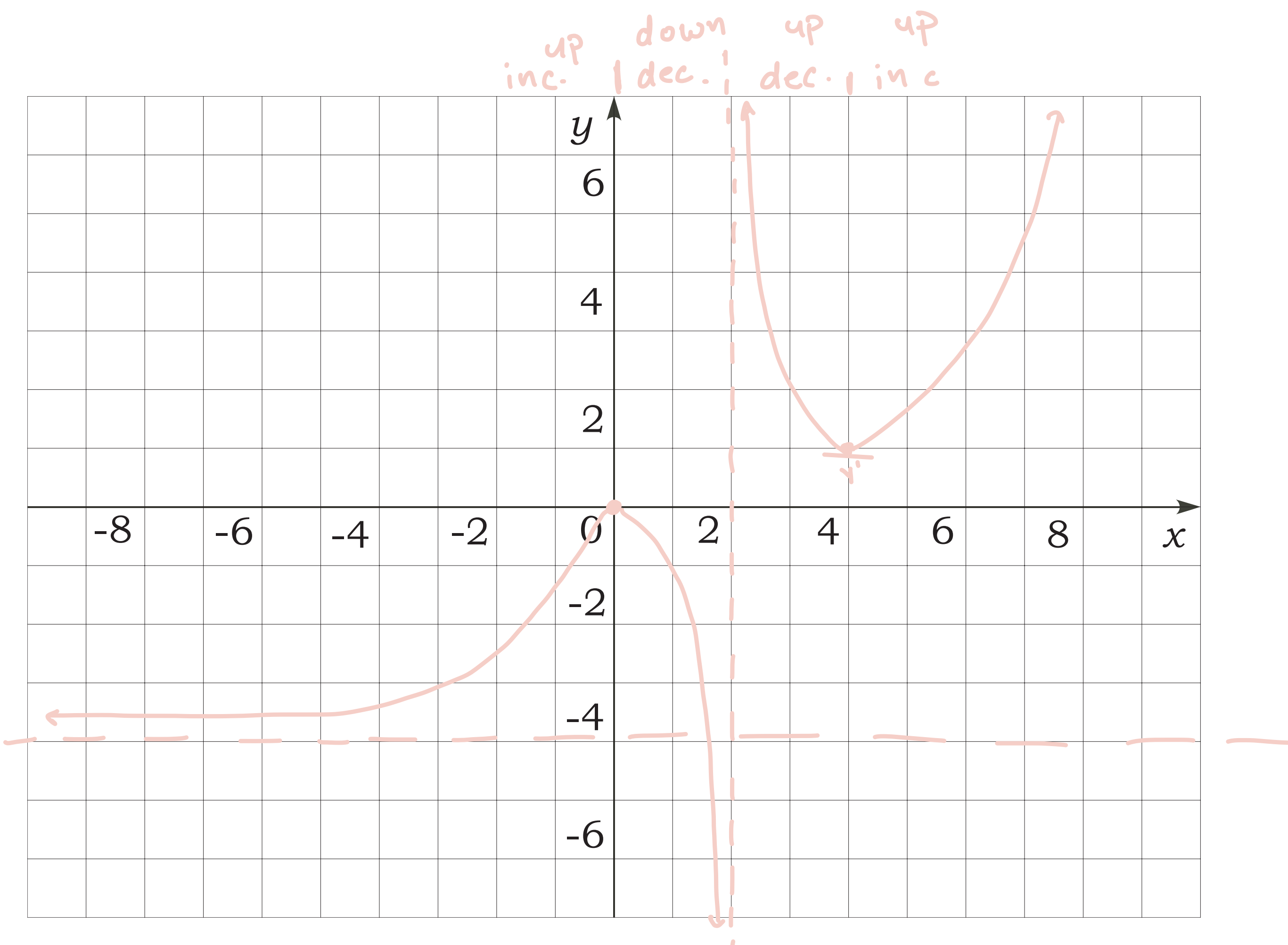
d. $y = -4$ is a horizontal asymptote.

e. $f'(x) > 0$ for $(-\infty, 0) \cup (4, \infty)$. inc.

f. $f'(x) < 0$ for $(0, 2) \cup (2, 4)$. dec.

g. $f''(x) > 0$ for $(-\infty, 0) \cup (2, \infty)$. up

h. $f''(x) < 0$ for $(0, 2)$. down



9. Find all vertical and horizontal asymptotes of the function $f(x) = \dfrac{e^{2x} + 5e^x - 6}{e^{2x} - 3e^x + 2}$.

let $u = e^x$

$$f(x) = \frac{u^2+5u-6}{u^2-3u+2}$$

$$= \frac{(u+6)\cancel{(u-1)}}{(u-2)\cancel{(u-1)}}$$

$$= \frac{e^x+6}{e^x-2}$$

horizontal:

$$\lim_{x\to\infty} \frac{e^x+6}{e^x-2} \quad \frac{\infty}{\infty} \quad \text{or} \cdot \frac{1/e^x}{1/e^x}$$

$$\text{L'H} = \lim_{x\to\infty} \frac{e^x}{e^x} \qquad = \lim_{x\to\infty} \frac{1+\frac{6}{e^x}}{1-\frac{2}{e^x}}$$

$$= \lim_{x\to\infty} 1 = 1 \qquad = \frac{1+0}{1-0} = \frac{1}{1} = 1$$

vertical: $e^x - 2 = 0$

$e^x = 2$

$\ln(e^x) = \ln(2)$

$x = \ln(2)$

$$\lim_{x\to-\infty} \frac{e^x+6}{e^x-2} = -3$$

think: what if I plug in -100

$$\Rightarrow \frac{e^{-100}+6}{e^{-100}-2} = \frac{\frac{1}{e^{100}}+6}{\frac{1}{e^{100}}-2} = \frac{6}{-2} = -3$$

10. Find the value(s) of c satisfying the conclusion of the MVT for function $f(x) = \sqrt{x-4}$ on the interval $[5, 13]$. Draw a picture to illustrate your answer.

$$\text{MVT: } f'(c) = \frac{f(b)-f(a)}{b-a}$$

$f(x) = (x-4)^{1/2}$

$f'(x) = \frac{1}{2}(x-4)^{-1/2}$

$f'(c) = \dfrac{1}{2\sqrt{c-4}}$

$f(b) = \sqrt{13-4} = \sqrt{9} = 3$

$f(a) = \sqrt{5-4} = \sqrt{1} = 1$

$$\frac{1}{2\sqrt{c-4}} = \frac{3-1}{13-5}$$

$$\frac{1}{2\sqrt{c-4}} = \frac{2}{8}$$

$$\frac{1}{2\sqrt{c-4}} = \frac{1}{4}$$

$4 = 2\sqrt{c-4}$

$2 = \sqrt{c-4}$

$4 = c-4$

$8 = c$

11. Find the range of the function $f(x) = 3 - xe^{-x^2/8}$ on $[1, \infty)$. State clearly if the function attains absolute minimum and absolute maximum on the given domain.

You may use the estimates $e^{-1/8} \approx 0.9$ and $e^{-1/2} \approx 0.6$.

$f(x) = 3 - \frac{x}{e^{x^2/8}}$

$f'(x) = (-1)e^{-x^2/8} + (-\frac{1}{4}xe^{-x^2/8})(-x)$

$= -e^{-x^2/8} + \frac{1}{4}x^2e^{-x^2/8}$

$0 = e^{-x^2/8}(-1 + \frac{1}{4}x^2)$

$0 = e^{-x^2/8}$ never

$0 = -1 + \frac{1}{4}x^2$

$1 = \frac{1}{4}x^2$

$4 = x^2$

$\pm 2 = x$

-2 is not in $[1, \infty]$

$[3 - 2e^{-1/2}, 3)$

$f(2) = 3 - \frac{2}{e^{(2)^2/8}}$

$= 3 - \frac{2}{e^{4/8}}$

$= 3 - \frac{2}{e^{1/2}}$

$\approx 3 - 2(0.6) = 0.8$

$f(1) = 3 - \frac{1}{e^{1^2/8}}$

$= 3 - \frac{1}{e^{1/8}}$

$\approx 3 - 0.9 = 2.1$

$\lim_{x \to \infty} 3 - \frac{x}{e^{x^2/8}}$

$= 3 - \lim_{x \to \infty} \frac{x}{e^{x^2/8}}$ $\frac{\infty}{\infty}$

L'H $= 3 - \lim_{x \to \infty} \frac{1}{\frac{1}{4}xe^{x^2/8}}$

$= 3 - 0$

12a. Find the dimensions of the rectangle with the smallest perimeter amongst all the rectangles with area 100 cm^2.

$A = x \cdot y = 100 \longrightarrow y = \frac{100}{x}$

$P = 2x + 2y$

$0 < x < \infty$

dimensions:

$x = 10$

$y = \frac{100}{10} = 10$

$P = 2x + 2\left(\frac{100}{x}\right)$

$= 2x + \frac{200}{x}$

$\frac{dP}{dx} = 2 - \frac{200}{x^2} \longrightarrow$ DNE when $x=0$ but $x=0$ is not in domain

$2 - \frac{200}{x^2} = 0$

$2 = \frac{200}{x^2}$

$2x^2 = 200$

$x^2 = 100$

$x = \pm 10 \longleftarrow$ toss out negative

$\lim_{x\to 0} 2x + \frac{200}{x} = 0 + \infty$

$\lim_{x\to \infty} 2x + \frac{200}{x} = \infty + 0$

$A(10) = 2(10) + \frac{200}{10}$

$= 20 + 20$

$= 40$

12b. Find the dimensions of the rectangle with the largest area amongst all the rectangles with perimeter 100 cm.

$P = 2x + 2y = 100 \longrightarrow y = \frac{100 - 2x}{2} = 50 - x$

$A = x \cdot y$

$= x(50 - x)$

$= 50x - x^2$

$0 \le x \le 50$

$\frac{dA}{dx} = 50 - 2x = 0$

$50 = 2x$

$25 = x$

$A(0) = 50(0) - (0)^2 = 0$

$A(50) = 50(50) - (50)^2 = 0$

$A(25) = 50(25) - (25)^2$

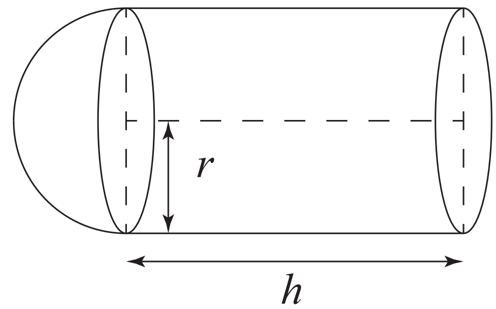
dimension

$x = 25$

$y = 50 - 25 = 25$

13. Consider a cylindrical container with a hemispherical cap on one end and a closed circular end as shown below. Let h be the height of the cylinder and r be its radius.

a. Find the volume V of the container in terms of the radius r only if the surface area of the container is 100π sq. meter. You may use the formulas $4\pi r^2$, $\frac{4}{3}\pi r^3$, $2\pi rh$ and $\pi r^2 h$.



Volume = hemisphere + cylinder

$= \frac{1}{2}(\frac{4}{3}\pi r^3) + \pi r^2 h$

SA = $\frac{1}{2}$(SA sphere) + bottom + sides

$100\pi = \frac{1}{2}(4\pi r^2) + (\pi r^2) + 2\pi rh$

$100\pi = 2\pi r^2 + \pi r^2 + 2\pi rh$

$100\pi = 3\pi r^2 + 2\pi rh$

$100\pi - 3\pi r^2 = 2\pi rh$

$\frac{100\pi - 3\pi r^2}{2\pi r} = h$

$\frac{50}{r} - \frac{3}{2}r = h$

$V = \frac{2}{3}\pi r^3 + \pi r^2 \left(\frac{50}{r} - \frac{3}{2}r\right)$

$= \frac{2}{3}\pi r^3 + 50\pi r - \frac{3}{2}\pi r^3$

$= \frac{4}{6}\pi r^3 + 50\pi r - \frac{9}{6}\pi r^3$

$= 50\pi r - \frac{5}{6}\pi r^3$

b. Find the possible values of r in the construction above.

You Are Not Required To Optimize the Function in Part (a).

Constraint:

r is a distance so it can't be negative

$100\pi = 3\pi r^2 + 2\pi rh$

when $r = 0$ the constraint is $100\pi = 3\pi(0)^2 + 2\pi(0)h = 0$

so r can not be zero

when r is really small ($r \to 0$)

when r is really large, h must be really small

$100\pi = 3\pi r^2 + 2\pi r(0)$

$100\pi = 3\pi r^2$

$\frac{100}{3} = r^2$

$\pm\sqrt{\frac{100}{3}} = r$

range of r is:

$0 < r \leq \frac{10}{\sqrt{3}}$