Math 10350 – Example Set 02B
Exponential & Logarithmic Function: Section 1.6

▶ Graph of $y = a^x$

Case 1: $a > 1$

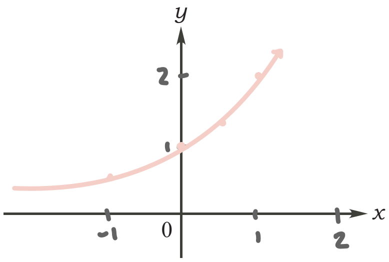
For example, $y = 2^x$.

(i) Complete the table below:

x	-1	-0.5	0	0.5	1
2^x	0.5	$2^{-1/2} = \frac{1}{\sqrt{2}}$	1	$2^{1/2} = \sqrt{2}$	2

Truncate answers to 2 decimal places

(ii) Plot the points and sketch graph:



(iii) **Properties of a^x when $a > 1$:**

- $a^0 \stackrel{?}{=} 1$
- domain $\stackrel{?}{=} (-\infty, \infty)$ $\mathbb{R}$ range $\stackrel{?}{=} (0, \infty)$
- $\lim\limits_{x \to -\infty} a^x \stackrel{?}{=} 0$ $\lim\limits_{x \to \infty} a^x \stackrel{?}{=} \infty$
- Asymptote: $y=0$

Case 2: $0 < a < 1$

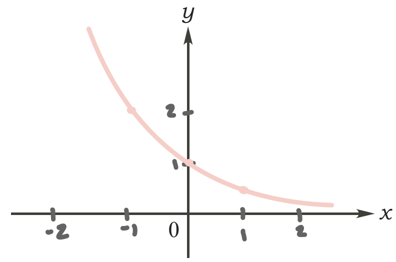
For example, $y = (1/2)^x$.

(i) Complete the table below:

x	-1	-0.5	0	0.5	1
$(1/2)^x$	2	$(\frac{1}{2})^{-1/2} = \sqrt{2}$	1	$(\frac{1}{2})^{1/2} = \sqrt{\frac{1}{2}}$	0.5

Truncate answers to 2 decimal places

(ii) Plot the points and sketch graph:



(iii) **Properties of a^x when $0 < a < 1$:**

- $a^0 \stackrel{?}{=} 1$
- domain $\stackrel{?}{=} (-\infty, \infty)$ range $\stackrel{?}{=} (0, \infty)$
- $\lim\limits_{x \to -\infty} a^x \stackrel{?}{=} \infty$ $\lim\limits_{x \to \infty} a^x \stackrel{?}{=} 0$
- Asymptote: $y=0$

1. Sketch the graph of the inverse of a^x for $a > 1$. State its domain and range. We call the inverse of a^x the logarithm function to the base a. Complete the following properties of the logarithm function to the base a below. Could you prove some of them?

i. Change to log base b: $\log_a x = \dfrac{\log_b(x)}{\log_b(a)}$

ii. $\log_a(a) \stackrel{?}{=} 1$

iii. $\log_a 1 \stackrel{?}{=} 0$

iv. $\log_a(xy) \stackrel{?}{=} \log_a(x) + \log_a(y)$

v. $\log_a(x^n) \stackrel{?}{=} n \cdot \log_a(x)$

vi. $\log_a \left(\dfrac{x}{y}\right) \stackrel{?}{=} \log_a(x) - \log_a(y)$

vii. $\log_a(a^x) \stackrel{?}{=} x \cdot \cancel{\log_a(a)} = x$

viii. $a^{\log_a x} \stackrel{?}{=} x$

Commonly used natural log properties: $\ln(e^x) \stackrel{?}{=} x$; $e^{\ln x} \stackrel{?}{=} x$

2. (Sect 1.6) A quantity y is said to grow or decay exponentially with time t if $y(t) = k \cdot a^t$. A . It is known that the amount of a medication in a patient reduces from an initial amount of 100 mg to 40 mg after three hours. Assuming that the amount of medication decays exponentially, write a formula for the amount of the medication $y(t)$ as a function of time t in hours. What is the half life of the medication in the body? Draw a graph for $y(t)$.

2. **(Sect 1.6)** A quantity y is said to grow or decay exponentially with time t if $y(t) = k \cdot a^t$. A . It is known that the amount of a medication in a patient reduces from an initial amount of 100 mg to 40 mg after three hours. Assuming that the amount of medication decays exponentially, write a formula for the amount of the medication $y(t)$ as a function of time t in hours. What is the half life of the medication in the body? Draw a graph for $y(t)$.

Step 1: Set up $y(t) = k \cdot a^t$

initial point: $y(0) = 100$

extra point: $y(3) = 40$

solve for k:

$100 = y(0) = k \cdot a^0$

$100 = k \cdot 1$

$100 = k$

solve for a:

$40 = y(3) = 100 \cdot a^3$

$\frac{4}{10} = a^3$

$\ln(\frac{4}{10}) = \ln(a^3)$ $\quad \ln(x^a) = a\ln(x)$

$\ln(\frac{4}{10}) = 3\ln(a)$

$\frac{1}{3}\ln(\frac{4}{10}) = \ln(a)$

$\cancel{\ln}((\frac{4}{10})^{1/3}) = \cancel{\ln}(a)$

$(\frac{4}{10})^{1/3} = a$

$y(t) = 100((\frac{4}{10})^{1/3})^t$

$y(t) = 100\,(\frac{2}{5})^{t/3}$

Step 2: Half Life

half life: the amount of time it takes to degrade to $\frac{1}{2}k$ i.e.

t s.t. $\frac{1}{2}k = ka^t \Rightarrow \frac{1}{2} = a^t$

$50 = 100\,(\frac{2}{5})^{t/3}$

$\frac{1}{2} = (\frac{2}{5})^{t/3}$

$\log_{\frac{2}{5}}(\frac{1}{2}) = \log_{\frac{2}{5}}(\frac{2}{5})^{t/3}$ $\quad \log(x^a) = a\log(x)$

$\log_{\frac{2}{5}}(\frac{1}{2}) = \frac{t}{3}\cancel{\log_{\frac{2}{5}}(\frac{2}{5})}$ $\quad \log_a a = 1$

$\log_{\frac{2}{5}}(\frac{1}{2}) = t/3$

$3\log_{\frac{2}{5}}(\frac{1}{2}) = t$

$\log_{\frac{2}{5}}(\frac{1}{2})^3 = t$

$\log_{\frac{2}{5}}(\frac{1}{8}) = t$

$2.27 \approx t$

using ln:

$\frac{1}{2} = (\frac{2}{5})^{t/3}$

$\ln(\frac{1}{2}) = \ln(\frac{2}{5})^{t/3}$

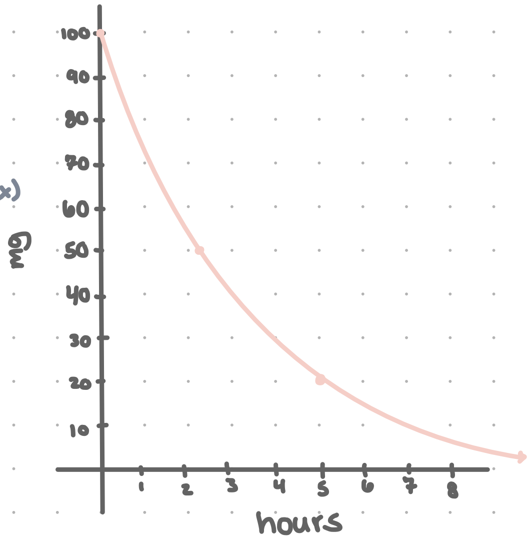
$\ln(\frac{1}{2}) = t\,\ln(\frac{2}{5})^{1/3}$

$t = \dfrac{\ln(1/2)}{\ln(2/5)^{1/3}}$

Step 3: Graph

$y(0) = 100$

$y(\sim 2.27) = 50$



3. Solve the following equations:

(a) $3(4^{x-1}) = 5$ (Give your answer in base e), **(b)** $4e^{x-2} = 3e^{2x}$

4. Given that $\ln(x) = p$ and $\ln(y) = q$, write the following expressions in term of p and q

(i) $\ln(5x^2y^3) \stackrel{?}{=}$ (ii) $\ln\left(\sqrt[5]{\frac{e^4x}{y}}\right) \stackrel{?}{=}$

3(a) $3(4^{x-1}) = 5$

$4^{x-1} = \frac{5}{3}$

$\ln(4^{x-1}) = \ln(\frac{5}{3})$ $\quad \ln(x^a) = a \cdot \ln(x)$

$(x-1)\ln(4) = \ln(\frac{5}{3})$

$x - 1 = \frac{\ln(\frac{5}{3})}{\ln(4)}$

$x = \frac{\ln(\frac{5}{3})}{\ln(4)} + 1$

(b) $4e^{x-2} = 3e^{2x}$

$\ln(4e^{x-2}) = \ln(3e^{2x})$ $\quad \ln(a \cdot b) = \ln(a) + \ln(b)$

$\ln(4) + \ln(e^{x-2}) = \ln(3) + \ln(e^{2x})$ $\quad \ln(x^a) = a \cdot \ln(x)$

$\ln(4) + (x-2)\cancel{\ln(e)} = \ln(3) + (2x)\cancel{\ln(e)}$ $\quad \log_a a = 1$

$\ln(4) + (x-2) = \ln(3) + 2x$

$-\ln(3) \quad -x \quad -\ln(3) \quad -x$

$\ln(4) - 2 - \ln(3) = x$ $\quad \ln(a) - \ln(b) = \ln(\frac{a}{b})$

$\ln(\frac{4}{3}) - 2 = x$

4(i) $\ln(5x^2y^3)$

Step 1: Separate $\quad \ln(a \cdot b) = \ln(a) + \ln(b)$

$\ln(5) + \ln(x^2) + \ln(y^3)$

Step 2: Remove powers $\quad \ln(x^a) = a \cdot \ln(x)$

$\ln(5) + 2\ln(x) + 3\ln(y)$

Step 3: Substitute $\quad p = \ln(x); q = \ln(y)$

$\ln(5) + 2p + 3q$

(ii) $\ln(\sqrt[5]{\frac{e^4x}{y}})$

$\ln((\frac{e^4x}{y})^{1/5})$ $\quad \sqrt[b]{x^a} = x^{\frac{a}{b}}$

$\frac{1}{5}\ln(\frac{e^4x}{y})$ $\quad \ln(x^a) = a \cdot \ln(x)$

$\frac{1}{5}[\ln(e^4x) - \ln(y)]$ $\quad \ln(\frac{a}{b}) = \ln(a) - \ln(b)$

$\frac{1}{5}[\ln(e^4) + \ln(x) - \ln(y)]$ $\quad \ln(a \cdot b) = \ln(a) + \ln(b)$

$\frac{1}{5}[4\ln(e) + \ln(x) - \ln(y)]$ $\quad \ln(x^a) = a \cdot \ln(x)$

$\frac{1}{5}[4 + \ln(x) - \ln(y)]$ $\quad \ln(e) = 1$

$\frac{4}{5} + \frac{1}{5}p + \frac{1}{5}q$ $\quad p = \ln(x); q = \ln(y)$