Math 10350 – Example Set 03A
Sections 2.1, 2.2, 2.3 & 2.4

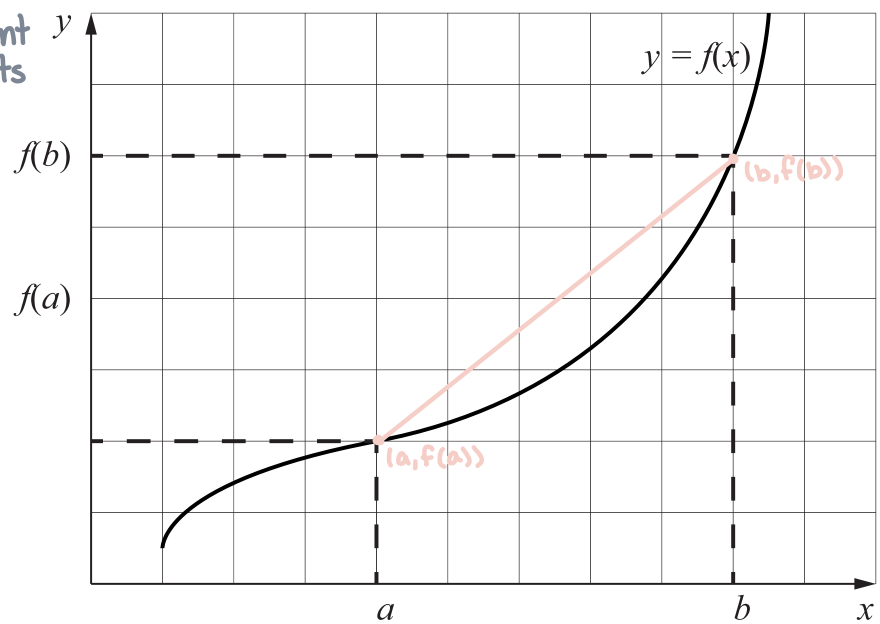
The Average Rate of Change of a function $f(x)$ over the interval $[a, b]$

is given by $= \dfrac{\text{Change in } f(x)}{\text{Change in } x} = \dfrac{\Delta f}{\Delta x} = \underline{\dfrac{f(x_2)-f(x_1)}{x_2-x_1}}$.

Sketch in the graph the chord (secant line) whose slope gives this average value.

chord: a line segment connecting 2 points on a curve

secant line: a line that intersects 2 points on a curve



slope of secant line: $m = \dfrac{f(b)-f(a)}{b-a}$

In the special case when the function is the position $s(t)$ meter of a particle moving on a straight line at time t seconds, the average rate of change of the position over the time interval $a \le t \le b$ is also called the

velocity over the time interval $a \le t \le b$

$= \dfrac{\text{Change in position}}{\text{Change in time}} = \dfrac{\Delta s}{\Delta t} = \underline{\dfrac{s(t_2)-s(t_1)}{t_2-t_1}}$ m/sec.

how to find units:

$s(t)$ = meters

t = seconds

$\dfrac{s(t_2)-s(t_1)}{t_2-t_1}$ $\dfrac{\text{meters}}{\text{seconds}}$

The Average Velocity and Instantaneous Velocity

1. The position (vertical height measured from the ground) of a ball projected vertically up from the ground is given by $s(t) = 30t - 5t^2$ meter at time t second. Find each of the following values and simplifying your answer.

(1a) Average rate of change of the position of the ball over the time interval $1 \le t \le 4$ = $\frac{40-25}{4-1} = \frac{15}{3} = 5$

|| special case

Average velocity of the ball over time interval $1 \le t \le 4 = \frac{\text{Change in position}}{\text{Change in time}} = \frac{40-25}{4-1} = \frac{15}{3} = 5\text{m/s}$

$s(4) = 30(4) - 5(4)^2 = 120 - 80 = 40$

$s(1) = 30(1) - 5(1)^2 = 30 - 5 = 25$

(1b) Average velocity over the time duration between 1 and t (assuming $t \ne 1$) =

$s(t) = 30t - 5t^2$

$s(1) = 30(1) - 5(1)^2 = 30 - 5 = 25$

$$\frac{s(t)-s(1)}{t-1} = \frac{30t-5t^2-25}{t-1} = \frac{-5(t^2-6t+5)}{t-1} = \frac{-5\cancel{(t-1)}(t-5)}{\cancel{t-1}}$$

$= -5(t-5)$

choose numbers as close to 1 as possible

(1c) Complete the table:

t	0.99	0.999	0.9999	1	1.0001	1.001	1.01
$\frac{s(t)-s(1)}{t-1}$	20.05	20.005	20.0005	?	19.9995	19.995	19.95

(1d) From the table, what could you observe about $\frac{s(t)-s(1)}{t-1}$? Give a physical interpretation for your observation.

where is the output heading as we approach 5 on both sides?

As t goes to 5, $\frac{s(t)-s(1)}{t-1}$ goes to 20

$\lim_{t \to 1} \frac{s(t)-s(1)}{t-1} = 20$ (mathematical statement)

the ball is moving 20 m/s at time t = 1 second

(1e) The above observation, we say that instantaneous velocity $v(1)$ of the ball at $t = 1$ second is given by the

limit of the average velocity $\frac{s(t)-s(1)}{t-1}$ as time t approaches 1.

prime

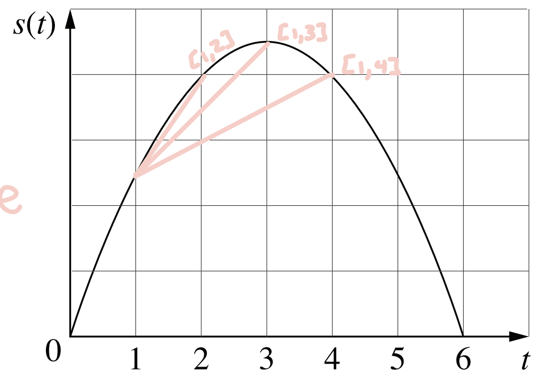
This denoted by $v(1) = s'(1) = \frac{ds}{dt} = \lim_{t \to 1} \frac{s(t)-s(1)}{t-1}$.

velocity

Δ denotes average

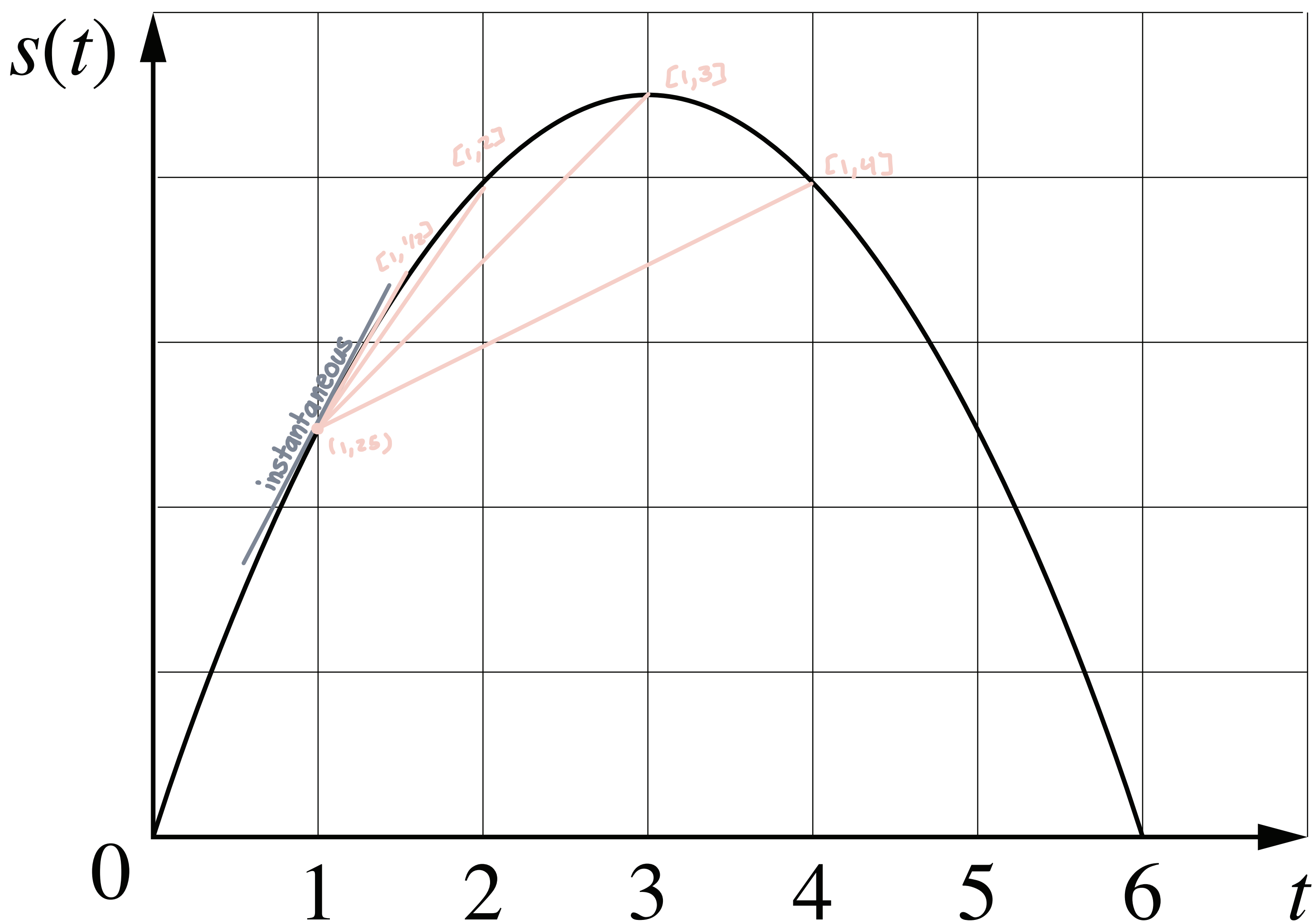
d denotes instantaneous

(1f) Give a graphical interpretation of the average velocity of the ball over the time interval $1 \le t \le 4$. Of course, we can also interpret the average velocity over the time interval between 1 and any $t (\neq 1)$.



On average, over the interval (1,4), the slope of the function is increasing

(1g) Give a graphical interpretation of the instantaneous velocity $v(1)$ of the ball at $t = 1$ second.



tangent line: a line that touches a curve at exactly one point

touches at t=1

(1h) Find the equation of the tangent line to the graph of $s(t) = 30 - 5t^2$ at $t = 1$.

$y - y_0 = m(x - x_0)$ where $(x_0, y_0) = (t, s(t))$ & m = instananeous velocity

$y - 25 = 20(x-1)$ $\quad (x_0, y_0) = (1, 25)$, $m = 20$

$y - 25 = 20x - 20$

$+25 \qquad +25$

$y = 20x + 5$

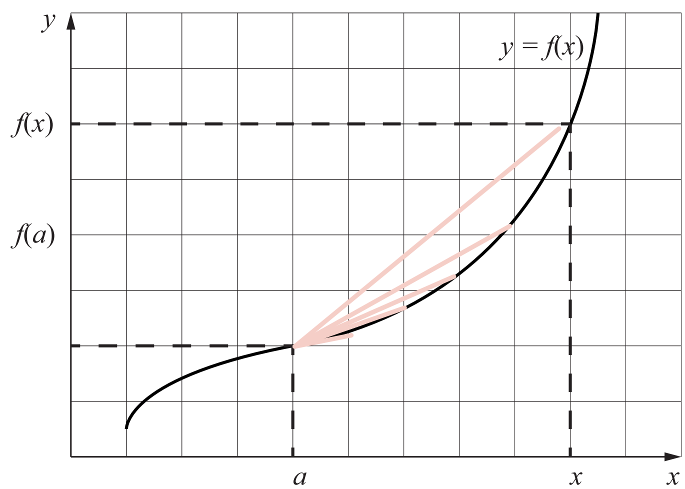
Summary. We have computed the instantaneous velocity at time $t = a$ of a particle moving on a straight line with position function $s(t)$. We online the key steps below.

Step 1: The average velocity of the particle over the time interval between t and a is $\frac{\Delta s}{\Delta t} = \frac{s(t)-s(a)}{t-a}$.

Step 2: The instantaneous velocity of the particle at $t = a$ is

$$v(a) = s'(a) = \frac{ds}{dt} = \lim_{t \to a} \frac{s(t)-s(a)}{t-a}$$

The same limiting process above can be applied to many functions besides the position function of a particle. We can mimic the same limiting process to find the **Instantaneous Rate of Change of a function** $f(x)$ **at a given** $x = a$**.** Illustrate the process in the graph below.



Step 1: The average rate of change of $f(x)$ over the time interval between x and a is $\frac{\Delta f}{\Delta x} = \frac{f(x)-f(a)}{x-a}$.

Step 2: The instantaneous rate of change of $f(x)$ at $x = a$ is

$$f'(a) = \frac{df}{dt} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$f'(a)$ is also called the derivative of f at the point a

and it gives the slope of the tangent line to the graph of $f(x)$ at $x = a$.

2. Water is flowing into a tank at a rate such that the volume $V(t)$ (in cubic feet) of water in the tank at time $t \geq 0$ (in minutes) is given by $V(t) = \sqrt{t+4}$. Answer the following questions:

(a) Find the average rate of change of the volume of water over the time duration $[5, 12]$. What is the unit of your answer?

(general formula)

average rate of change $= \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$ $\quad \dfrac{\text{cubic feet}}{\text{minute}}$

over $[5,12] = \dfrac{\sqrt{12+4}-\sqrt{5+4}}{12-5} = \dfrac{\sqrt{16}-\sqrt{9}}{7} = \dfrac{4-3}{7} = \dfrac{1}{7}$ ft³/m

(b) Using limits, find the rate of change of the volume of water at the fifth minute. What is the unit of your answer?

(general formula)

instantaneous rate of change $= \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$

at $5 = \lim\limits_{x \to 5} \dfrac{f(x)-f(5)}{x-5}$

$= \lim\limits_{x \to 5} \dfrac{\sqrt{x+4}-\sqrt{5+4}}{x-5}$

$= \lim\limits_{x \to 5} \dfrac{\sqrt{x+4}-\sqrt{9}}{x-5}$

$f'(5) = \lim\limits_{x \to 5} \dfrac{\sqrt{x+4}-3}{x-5}$

As of right now, we do not know any limit rules or derivative rules to help us, so we must utilize a chart to find the limit →

x	4.9	4.99	4.999	5	5.001	5.01	5.1
$\frac{f(x)-f(a)}{x-a}$	0.167	0.1667	0.16667	?	0.16666	0.1666	0.166

$f'(5) = 0.1\overline{66} = \frac{1}{6}$