Math 10350 – Example Set 03B
Sections 2.2, 2.3 & 2.4

Limit of a function. What happens to $f(x)$ as x gets as close to a fixed value c as we want? This question is answered with the concept of the limit of a function.

Explain what each of the following limits mean.

$L = \lim_{x \to c^-} f(x)$ where f(x) goes as we increase x to c

We call this the left-handed limit of $f(x)$ as x approaches c.

$L = \lim_{x \to c^+} f(x)$ where f(x) goes as we decrease x to c

We call this the right-handed limit of $f(x)$ as x approaches c.

$L = \lim_{x \to c} f(x)$ where f(x) goes as x goes to c on both sides

We call this the (two-handed) limit of $f(x)$ as x approaches c.
(two-sided)

1. The graph of a function f is shown in Figure 1. By inspecting the graph, find each of the following values and limits if it exists. If the limit does not exist, explain why.

$\lim_{x \to -1} f(x) \stackrel{?}{=}$ 1 $\qquad f(-1) \stackrel{?}{=}$ 3

$\lim_{x \to -1^-} f(x) \stackrel{?}{=}$ 1 $\qquad \lim_{x \to -1^+} f(x) \stackrel{?}{=}$ 1

$\lim_{x \to 0} f(x) \stackrel{?}{=}$ DNE $\qquad f(0) \stackrel{?}{=}$ 2

$\lim_{x \to 0^-} f(x) \stackrel{?}{=}$ 2 $\qquad \lim_{x \to 0^+} f(x) \stackrel{?}{=}$ 3

$\lim_{x \to 2} f(x) \stackrel{?}{=}$ DNE $\qquad f(2) \stackrel{?}{=}$ 3

$\lim_{x \to 2^-} f(x) \stackrel{?}{=} \infty$ $\qquad \lim_{x \to 2^+} f(x) \stackrel{?}{=}$ 3

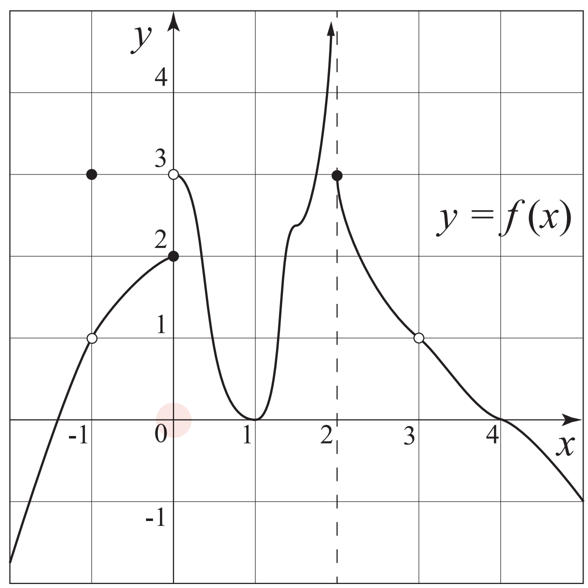

Figure 1

$\lim_{x \to 3} f(x) \stackrel{?}{=}$ 1 $\qquad f(3) \stackrel{?}{=}$ DNE

$\lim_{x \to 3^-} f(x) \stackrel{?}{=}$ 1 $\qquad \lim_{x \to 3^+} f(x) \stackrel{?}{=}$ 1

$\boxed{\textbf{Theorem 1}}$ (1) $\lim_{x\to c} f(x)$ exists $\iff$ $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ both exist and are equal.

Moreover, (2) $\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)$.

$\boxed{\textbf{Remark}}$ If any of the following are true:

$$\lim_{x\to c^-} f(x) = \infty; \qquad \lim_{x\to c^-} f(x) = -\infty; \qquad \lim_{x\to c^+} f(x) = \infty; \qquad \lim_{x\to c^+} f(x) = -\infty$$

then the graph of $f(x)$ has an asymptote at $x = c$.

$\boxed{\textbf{Definition}}$ (a) A function $f(x)$ is continuous at $x = c \iff f(c)$ is defined and $\lim_{x\to c} f(x) = f(c)$.

(b) A function $f(x)$ is **left** continuous at $x = c \iff f(c)$ is defined and $\lim_{x\to c^-} f(x) = f(c)$.

(c) A function $f(x)$ is **right** continuous at $x = c \iff f(c)$ is defined and $\lim_{x\to c^+} f(x) = f(c)$.

(d) A function $f(x)$ has a **jump** discontinuous at $x = c \iff \lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$.

(e) A function $f(x)$ has a **removable** discontinuous at $x = c \iff \lim_{x\to c} f(x)$ exist but $\lim_{x\to c} f(x) \neq f(c)$.

also called a hole

2. Comment on the continuity at $x = -1, 0, \cancel{1}\,3, 2$ for $f(x)$ in Figure 1. Are there any removable discontinuity?

I. $x=-1$

$\lim_{x\to -1} f(x) = 1$, but $f(-1)=3$

removeable discontinuity

II. $x=0$

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

jump discontinuity

III. $x=2$

$\lim_{x\to 2^-} f(x) = \infty \neq \lim_{x\to 2^+} f(x)$

asymptote & jump discontinuity

IV. $x=3$

$\lim_{x\to 3} f(x) = 1$, $f(3)$ DNE

removeable discontinuity

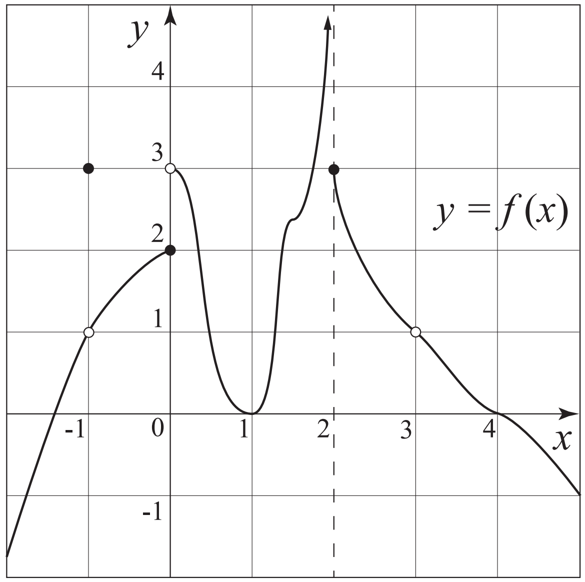

Figure 1

3. The graph of $f(x)$ is given in Figure 1 and $g(x) = 3x + 2$. By thinking about the values each function $f(x)$ and $g(x)$ approaches in the expressions below deduce the value of each limits:

(a) $\lim_{x\to 3} [2f(x) + 3g(x)] \overset{?}{=}$

$= 2\lim_{x\to 3} f(x) + 3\lim_{x\to 3} g(x)$

$= 2(1) + 3(3(3)+2))$

$= 2 + 3(11)$

$= 2 + 33$

$= 35$

(d) $\lim_{x\to 0} [f(x) - g(x)]^4 \overset{?}{=}$

$= [\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)]^4$

$= [(DNE) - (3(0)+2)]^4$

$= DNE$

(b) $\lim_{x\to 2^+} [f(x) \cdot g(x)] \overset{?}{=}$

$= \lim_{x\to 2^+} f(x) \cdot \lim_{x\to 2^+} g(x)$

$= 3 \cdot (3(2)+2)$

$= 3 \cdot 8$

$= 24$

(e) $\lim_{x\to 0} \sqrt{f(x)} \overset{?}{=}$

$= \sqrt{\lim_{x\to 0} f(x)}$

$= \sqrt{DNE}$

$= DNE$

(c) $\lim_{x\to 0^-} \dfrac{g(x)}{f(x)+4} \overset{?}{=}$

$= \dfrac{\lim_{x\to 0^-} g(x)}{\lim_{x\to 0^-} (f(x))+4}$

$= \dfrac{3(0)+2}{2+4}$

$= \dfrac{2}{6} = \dfrac{1}{3}$

Properties of Limits. Suppose $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist. Then we have the following statements:

(1) $\lim_{x \to c} k \cdot f(x) =$

$k \cdot \lim_{x \to c} f(x)$

(2) $\lim_{x \to c} [f(x) + g(x)] =$

$\lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

(3) $\lim_{x \to c} [f(x) - g(x)] =$

$\lim_{x \to c} f(x) - \lim_{x \to c} g(x)$

(4) $\lim_{x \to c} [f(x) \cdot g(x)] =$

$\lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

(5) $\lim_{x \to c} \frac{f(x)}{g(x)} =$ provided $\lim_{x \to c} g(x) \neq 0$.

$\frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$

(6) $\lim_{x \to c} [f(x)]^n =$

$\left[\lim_{x \to c} f(x)\right]^n$

(7) $\lim_{x \to c} \sqrt[n]{f(x)} =$ $\lim_{x \to c} f(x) \geq 0$ if n is even.

$\left[\lim_{x \to c} f(x)\right]^{1/n}$

$\sqrt[n]{\lim_{x \to c} f(x)}$