Math 10350 – Example Set 03C
Sections 2.2, 2.3 & 2.4
Computing Limits and Continuity

1. (Limit Concept) Consider the **piecewise defined** function
$$f(x) = \begin{cases} 2-x & -\infty < x < 0 \\ x^2 & 0 \le x < 1 \\ 1 & 1 \le x < +\infty \end{cases}$$

$2-x$ | x^2 | 1
section I | 0 | II | 1 | III

(a) Compute the following limits if they exist without sketching the graph.

i. $f(-1)$ **ii.** $f(3)$ **iii.** $\lim_{x\to 1^-} [2f(x) - 5]$ **iv.** $\lim_{x\to 0} (f(x) - 1)^2$ **v.** $\lim_{x\to -5} (f(x))^2$

(i) $f(-1)$
$-1<0$
section I
$f(-1) = 2-(-1)$
$= 2+1$
$= 3$

(ii) $f(3)$
$3 \ge 1$
section III
$f(3) = 1$

(iii) $\lim_{x\to 1^-}[2f(x)-5]$
$= 2\underbrace{\lim_{x\to 1^-} f(x)}_{\text{section II}} - 5$
$\lim_{x\to 1^-} x^2 = 1$
$= 2(1) - 5$
$= 2-5$
$= -3$

(iv) $\lim_{x\to 0}(f(x)-1)^2$
$= [\lim_{x\to 0}(f(x)-1)]^2$
$= [\underbrace{\lim_{x\to 0} f(x)}_{\text{section I \& II}} - 1]^2$
$\lim_{x\to 0^-} 2-x = 2$
$\lim_{x\to 0^+} x^2 = 0$
$>$DNE
$=$ DNE

(v) $\lim_{x\to -5}(f(x))^2$
$= [\underbrace{\lim_{x\to -5} f(x)}_{\text{section I}}]^2$
$\lim_{x\to -5}(2-x) = 2-(-5)$
$= 7$
$= [7]^2$
$= 49$

(b) Without sketching the graph of $f(x)$, determine the continuity of $f(x)$ at (i) $x = 0$ and (ii) $x = 1$.

(i) $x=0$
$\lim_{x\to 0^-} f(x)$
section I
$= \lim_{x\to 0^-}(2-x)$
$= 2$

$\lim_{x\to 0^+} f(x)$
$= \lim_{x\to 0^+}(x^2)$
section II
$= 0$

$f(0) = (0)^2$
section II
right continuous
jump discontinuity

(ii) $x = 1$
$\lim_{x\to 1^-} f(x)$
section II
$= \lim_{x\to 1^-}(x^2)$
$= 1$

$\lim_{x\to 1^+} f(x)$
section III
$= \lim_{x\to 1^+}(1)$
$= 1$

$f(1) = 1$
section III
continuous

(c) Sketch the graph of $f(x) = \begin{cases} 2 - x & -\infty < x < 0 \\ x^2 & 0 \le x < 1 \\ 1 & 1 \le x < +\infty \end{cases}$

section I
section II
section III

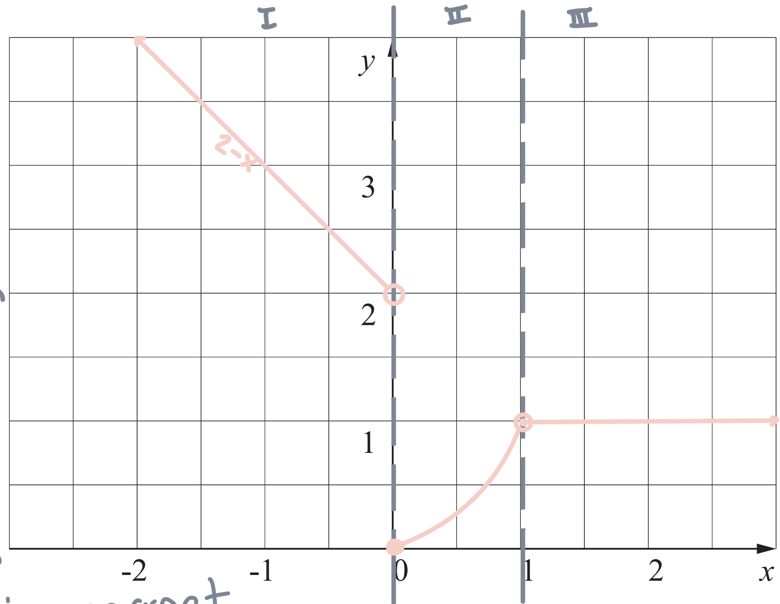
How to graph piece-wise function:

$f(x) = \begin{cases} f_1(x) & a < x \le b \\ f_2(x) & c < x < d \\ f_3(x) & e \le x < f \end{cases}$

Step 1: plot the "edge" points (a,b,c,d,e,f) with open circles

Step 2: fill in open holes where there are equal to signs

Step 3: connect using correct function for each section



Step 1:

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2-x)$
$= 2$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2)$
$= 0$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2)$
$= 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1)$
$= 1$

open circle at (0,2), (0,0), (1,1)

Step 2:

$f(0) = (0)^2$
$= 0$

$f(1) = 1$

fill in circles (0,0), (1,1)

Step 3:

- graph $f(x) = 2 - x$ in section 1
- graph $f(x) = x^2$ in section 2
- graph $f(x) = 1$ in section 3

2. Using limits find k such that the function $f(x)$ (a) is continuous at $x = 2$ and (b) is discontinuous at $x = 2$. Sketch a graph clearly depicting the nature of $f(x)$ at $x = 2$ in (a) and (b).

graphing first reveals that this is a hole & you can visually see what k should be

$$f(x) = \begin{cases} \dfrac{x^2+x-6}{x-2} & x \neq 2 \\ k & x = 2 \end{cases}$$

any time x=c is singled out (with x≠c as the other section) we have a hole

To be continuous at x=2, we need to fill the hole that way $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$. To create a discontinuity we must break one equality.

(a) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2+x-6}{x-2}$

$= \lim_{x\to 2^-} \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}}$

$= \lim_{x\to 2^-} x+3$

$= 5$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2+x-6}{x-2}$

$= \lim_{x\to 2^+} \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}}$

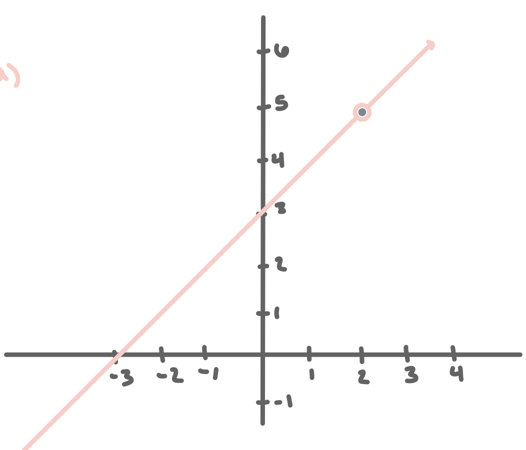
$= \lim_{x\to 2^+} x+3$

$= 5$

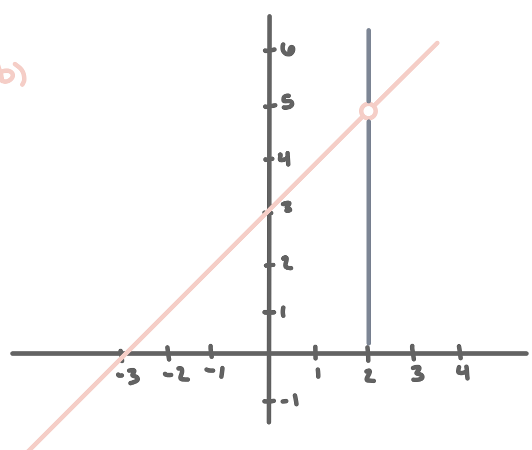
to fill the hole we need $f(2)=5$, thus $k=5$

(b) To leave the discontinuity, we make k anything but 5

(a)



(b)



3. Determine the constants a and b such that the following function is continuous on the entire real line

$$f(x) = \begin{cases} 2 & -\infty < x \leq -1 \\ ax^2 + b & -1 < x < 3 \\ -2 & 3 \leq x < +\infty \end{cases}$$

To be continuous, we need each of the pieces to line up, i.e. $\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$ for all a

Fix hole at $x=-1$

$\lim_{x\to -1^-} f(x) = f(-1) = \lim_{x\to -1^+} f(x)$

section I I II

$\lim_{x\to -1^-} (2) = 2 = \lim_{x\to -1^+} (ax^2+b)$

$2 = \lim_{x\to -1^+} (ax^2+b)$

$2 = a+b$

Fix hole at $x=3$

$\lim_{x\to 3^-} f(x) = f(3) = \lim_{x\to 3^+} f(x)$

section II III III

$\lim_{x\to 3^-} (ax^2+b) = -2 = \lim_{x\to 3^+} (-2)$

$\lim_{x\to 3^-} (ax^2+b) = -2$

$a(3)^2 + b = -2$

$9a+b = -2$

We must now reconcile these two equations:

(i) $a+b=2$

(ii) $9a+b=-2$

⇓ step2

(i) $a=2-b$

(ii) $9a+b=-2$

⇓ step3

(i) $a=2-b$

(ii) $9(2-b)+b=-2$

Step 4
$18-9b+b=-2$
$18-8b=-2$
$-8b=-20$
$b=\frac{-20}{-8}$
$b=\frac{5}{2}$

⇓ step5

(i) $a = 2-\frac{5}{2}$

$a = \frac{4}{2}-\frac{5}{2}$

$a=-\frac{1}{2}$

we use the method of substitution to solve:

Step 1: label the equations

Step 2: solve eq. (i) for a

Step 3: plug this into eq. (ii)

Step 4: solve eq. (ii) for b

Step 5: plug b back into eq. (i)

check:

$\lim_{x\to -1^+} -\frac{1}{2}x^2+\frac{5}{2} = -\frac{1}{2}(-1)^2+\frac{5}{2}$
$= -\frac{1}{2}+\frac{5}{2}$
$= \frac{4}{2} = 2$

$\lim_{x\to 3^-} -\frac{1}{2}x^2+\frac{5}{2} = -\frac{1}{2}(3)^2+\frac{5}{2}$
$= -\frac{9}{2}+\frac{5}{2}$
$= -\frac{4}{2} = -2$

$\boxed{\begin{array}{l} f(x) = ax^2+b \\ f(x) = -\frac{1}{2}x^2+\frac{5}{2} \end{array}}$