Introduction to derivatives

Up until now, we have defined instaneous rate of change using the formula:
$\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

This utilizes the formula for the average rate of change over [x,c] and sees what happens when x approach c, but doesn't quite reach it.

There is another way to consider this value. Rather that trying to bring x close to c, we can consider the range [x,x+h] where h goes to zero. The idea is to decrease the change of x to be close to nothing.

We now use an updated instantaneous rate of change formula that does not depend on a value c:
$\lim_{h \to 0} \frac{f(x+h)-f(x)}{x+h-x} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Example. Let $f(x)=\frac{1}{x^2}$. Find the instantaneous rate of change at $x=2$ both ways.

(a) old formula: $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

when $f(x)=\frac{1}{x^2}, c=2$: $\lim_{x \to 2} \frac{\frac{1}{x^2}-\frac{1}{4}}{x-2}$

$= \lim_{x \to 2} \frac{\frac{4-x^2}{4x^2}}{x-2}$

$= \lim_{x \to 2} \frac{\frac{(2-x)(2+x)}{4x^2}}{\frac{x-2}{1}}$

$= \lim_{x \to 2} \frac{(2-x)(2+x)}{4x^2} \cdot \frac{1}{x-2}$

$= \lim_{x \to 2} \frac{\cancel{(2-x)}(2+x)}{4x^2} \cdot \frac{1}{-1\cancel{(-x+2)}}$

$= \lim_{x \to 2} \frac{2+x}{-4x^2}$

$= \frac{2+2}{-4(2)^2}$

$= \frac{4}{-16}$

$= -\frac{1}{4}$

(b) new formula: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

when $f(x)=\frac{1}{x^2}$: $\lim_{h \to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

$= \lim_{h \to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$

$= \lim_{h \to 0} \frac{\frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}}{h}$

$= \lim_{h \to 0} \frac{\frac{\cancel{x^2}-\cancel{x^2}-2xh-h^2}{x^2(x+h)^2}}{h}$

$= \lim_{h \to 0} \frac{\frac{-2xh-h^2}{x^2(x+h)^2}}{\frac{h}{1}}$

$= \lim_{h \to 0} \frac{-2xh-h^2}{x^2(x+h)^2} \cdot \frac{1}{h}$

$= \lim_{h \to 0} \frac{-\cancel{h}(2x+h)}{x^2(x+h)^2} \cdot \frac{1}{\cancel{h}}$

$= \lim_{h \to 0} \frac{-(2x+h)}{x^2(x+h)^2}$

$= \frac{-2x}{x^2 \cdot x^2}$

$= \frac{-2}{x^3}$

$f'(2) = \frac{-2}{(2)^3} = -\frac{2}{8} = -\frac{1}{4}$

Math 10350 – Example Set 04A
Sections 3.1& 3.2
Differentiability & Derivative of a Function

Definition 1 A function $f(x)$ is said to be <u>differentiable</u> at $x = c$ provided the following limit exist:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Rather than finding the instantaneous rate of change at a point $x = c$ using $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$. We can find the instantaneous rate for any point in the domain of $f(x)$ and then plug in c. To do that we use the formula $\lim_{h \to 0} \frac{f(x+h) - f(x)}{x+h - x}$. When $x = c$, $\lim_{h \to 0} \frac{f(c+h) - f(c)}{c+h-c}$.

This means that the slope at $x = c$ of the graph is a ___real___ number. We denote this number by $f'(c)$.

Graphically, differentiable means that each small segment of the graph of $f(x)$ is almost identical to a straight line. This is illustrated in Figure 1 through 3 below. As you zoom into the point $(c, f(c))$, the segment of the graph of $f(x)$ near point c becomes more and more like its tangent line at $x = c$.

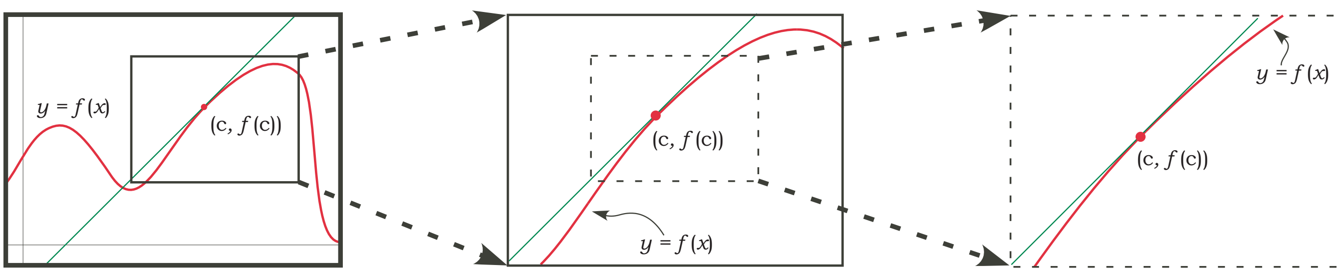


Figure 1. Figure 2. Figure 3.

Remark: We say that a function $f(x)$ is differentiable on (a, b) if $f(x)$ is differentiable for all $x = c$ in (a, b).

Theorem 1 If $f(x)$ be <u>differentiable</u> at $x = c$, then $f(x)$ is ___continuous___ at $x = c$.

1. Let $f(x) = \dfrac{1}{x^2}$. Compute the derivative or the slope function of $f(x)$ using limits by following steps below.

a. Find the average rate of change of $f(x)$ over the interval between x and $x + h$ assuming that $h \neq 0$.

This is also called the ____________________

b. Using (a), find the derivative of $f(x)$ (w.r.t. x) using the limit definition.

c. What is the instantaneous rate of change of $f(x)$? ____________________

d. Find the equation of the tangent line to the graph of $f(x)$ at $x = 2$. Draw a graph that describe the limiting process in (c) and its connection to the tangent line.

1. Let $f(x) = \dfrac{1}{x^2}$. Compute the derivative or the slope function of $f(x)$ using limits by following steps below.

a. Find the average rate of change of $f(x)$ over the interval between x and $x+h$ assuming that $h \neq 0$.

(a) general expression: $\dfrac{f(x+h)-f(x)}{x+h-x} = \dfrac{f(x+h)-f(x)}{h}$

when $f(x)=\frac{1}{x^2}$: $\dfrac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \dfrac{\frac{x^2-(x+h)^2}{(x+h)^2x^2}}{h} = \dfrac{\frac{\cancel{x^2}-\cancel{x^2}-2hx-h^2}{x^2(x+h)^2}}{h} = \dfrac{\frac{-2hx-h^2}{x^2(x+h)^2}}{h}$

multiply by reciprocal $= \dfrac{-2hx-h^2}{x^2(x+h)^2}\cdot\dfrac{1}{h} = \dfrac{\cancel{h}(-2x-h)}{\cancel{h}x^2(x+h)^2} = \dfrac{-2x-h}{x^2(x+h)^2}$

b. Using (a), find the derivative of $f(x)$ (w.r.t. x) using the limit definition.

(b) general expression: $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{x+h-x} = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

when $f(x)=\frac{1}{x^2}$: $\lim\limits_{h\to 0}\dfrac{-2x-h}{x^2(x+h)^2} = \dfrac{-2x}{x^4} = \dfrac{-2}{x^3}$

c. What is the instantaneous rate of change of $f(x)$? ________________________

(c)

d. Find the equation of the tangent line to the graph of $f(x)$ at $x=2$. Draw a graph that describe the limiting process in (c) and its connection to the tangent line.

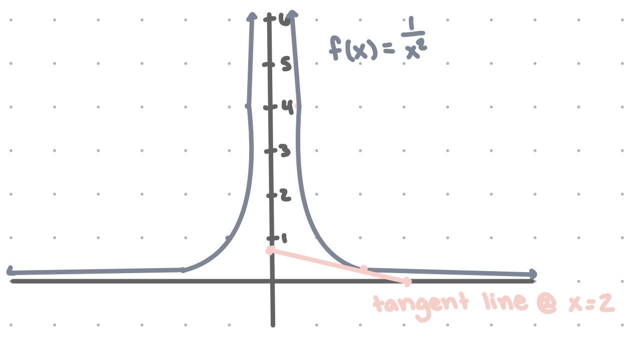
(d) general tangent line equation at $x=c$: $y - f(c) = f'(c)(x-c)$

when $f(x)=\frac{1}{x^2}$ & $x=2$: $y - f(2) = f'(2)(x-2)$ $\quad f(2) = \frac{1}{2^2} = \frac{1}{4}$

$y - \frac{1}{4} = \frac{-1}{4}(x-2)$ $\quad f'(2) = \frac{-2}{2^3} = \frac{-1}{2^2} = \frac{-1}{4}$

$y - \frac{1}{4} = \frac{-1}{4}x + \frac{2}{4}$

$y = \frac{-1}{4}x + \frac{3}{4}$



Derivative of a function. The derivative of the function $f(x)$ is given by the following limit:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Setting $\Delta x = h$ and $\Delta y = f(x+h) - f(x)$ gives the notation:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{x+h-x} = \lim \frac{f(x+h)-f(x)}{h}$$

Notation: If $y = f(x)$ is a differentiable function. Write down all standard notations of the derivative of $y = f(x)$.

$\frac{df}{dx} = f'(x) = \frac{d}{dx}(f)$ or $\frac{dy}{dx} = y'(x) = \frac{d}{dx}(y)$

Some Common Derivatives. For any numbers k and n:

$\frac{d}{dx}(k) \stackrel{?}{=} \lim_{h \to 0} \frac{k-k}{h} = 0$ $\qquad$ $\frac{d}{dx}(x^n) \stackrel{?}{=} nx^{n-1}$ (Power Rule)

Basic Properties of Derivatives:

$[f(x) + g(x)]' \stackrel{?}{=} f'(x) + g'(x)$ $\qquad$ $[f(x) - g(x)]' \stackrel{?}{=} f'(x) - g'(x)$ $\qquad$ $[c \cdot f(x)]' \stackrel{?}{=} c \cdot f'(x)$

2. Find the derivative of each of the following functions with respect to the :

a. $f(x) = \sqrt{x} + \frac{\pi}{\sqrt{x}}$

$= (x)^{1/2} + \pi x^{-1/2}$

$= \frac{1}{2}x^{-1/2} + -\frac{1}{2}\pi x^{-3/2}$

$= \frac{1}{2\sqrt{x}} - \frac{\pi}{2\sqrt[2]{x^3}}$

b. $y = \frac{x^3 + 5x + 6}{x}$

$= \frac{x^3}{x} + \frac{5x}{x} + \frac{6}{x}$

$= x^2 + 5 + 6x^{-1}$

$= 2x + 0 - 6x^{-2}$

$= 2x - \frac{6}{x^2}$

c. $h(t) = \left(2 + \sqrt{t}\right) t^2$

$= 2t^2 + t^2 t^{1/2}$

$= 2t^2 + t^{5/2}$

$= 4t + \frac{5}{2}t^{3/2}$ $\qquad$ $\frac{5}{2} - \frac{2}{2} = \frac{3}{2}$