Math 10350 – Example Set 05B
Section 3.5 Higher Derivatives
Section 3.6 Trigonometric Functions

1. Consider the function $f(t) = t^4 - 2e^t + 2$.

a. Find the following derivatives of $f(t)$: (i) $f'(t)$, (ii) $f''(t) = \dfrac{d^2 f}{dt^2}$, (iii) $f'''(t)$, and (iv) $\dfrac{d^4 f}{dt^4}$.

b. If $f(t)$ represents the position of a particle moving on a straight line, what would $f'(t)$ and $f''(t)$ mean physically?

2. Define the trigonometric functions:

$$\tan x = \frac{\sin x}{\cos x}\ , \quad \cot x = \frac{\cos x}{\sin x}\ , \quad \sec x = \frac{1}{\cos x} \quad \text{and} \quad \csc x = \frac{1}{\sin x}.$$

Use the fact that $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$ to show that

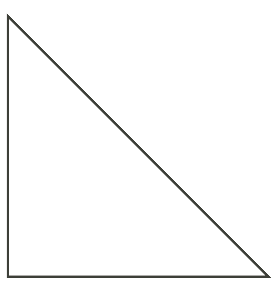
a. $\dfrac{d}{dx}(\tan x) = \sec^2 x$

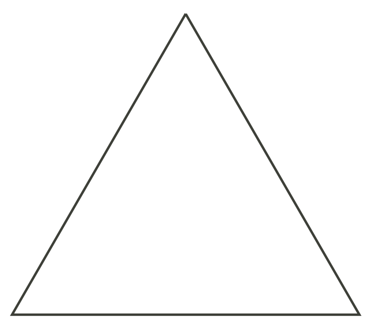
b. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$

c. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

d. $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

3. Using the equilateral triangle and right isosceles triangle, determine all trigonometric ratios of the special angles $\pi/6$, $\pi/4$ and $\pi/3$.





4. A piece if wood floating on the surface of a pond is bobbing up and down according to the position function

$$s(t) = \cos(t) + \sin(t) \quad \text{cm}$$

where t is in seconds.

(a) Find formulas for both its velocity and acceleration at time t seconds.

(b) Find smallest time at which the velocity of the piece of wood is zero.

5. Assuming that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x} = 0$, answer the questions below:

a. Find the values of (i) $\displaystyle\lim_{x\to 0} \frac{\sin 7x}{3x}$ and (ii) $\displaystyle\lim_{x\to 0} \frac{\tan x}{2x}$

b. Show that the derivative of $\sin x$ is $\cos x$. You will need the identity $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

1. Consider the function $f(t) = t^4 - 2e^t + 2$.

a. Find the following derivatives of $f(t)$: (i) $f'(t)$, (ii) $f''(t) = \frac{d^2 f}{dt^2}$, (iii) $f'''(t)$, and (iv) $\frac{d^4 f}{dt^4}$.

Derivative notations: $f'(t), \frac{d}{dt}f$ Second Derivative: $f''(t), (\frac{d}{dt})^2 f = \frac{d^2}{dt^2} f$ etc.

(a) i. $f'(t) = 4t^3 - 2e^t + 0$
ii. $f''(t) = 12t^2 - 2e^t$
iii. $f'''(t) = 24t - 2e^t$
iv. $f''''(t) = 24 - 2e^t$

derivative rules:
$f(x) = c \Rightarrow f'(x) = 0$
$f(x) = ax^n \Rightarrow f'(x) = a \cdot n \cdot x^{n-1}$
$f(x) = ae^x \Rightarrow f'(x) = ae^x$

b. If $f(t)$ represents the position of a particle moving on a straight line, what would $f'(t)$ and $f''(t)$ mean physically?

If $f(t) = s(t)$ the position function then $f'(t) = s'(t)$ is the instaneous rate of change of position, also known as velocity $v(t)$. Which makes $f''(t) = s''(t) = v'(t)$ the rate of change of velocity, i.e. the acceleration function.

2. Define the trigonometric functions:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x} \quad \text{and} \quad \csc x = \frac{1}{\sin x}.$$

Use the fact that $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ to show that

a. $\frac{d}{dx}(\tan x) = \sec^2 x$

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c. $\frac{d}{dx}(\sec x) = \sec x \tan x$

d. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

(a) $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$= \frac{(\cos x)(\cos x) - (-\sin x)(\sin x)}{(\cos x)^2}$

quotient rule: $\frac{f'g - g'f}{g^2}$
$f = \sin x \quad g = \cos x$
$f' = \cos x \quad g' = -\sin x$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x}$ $\quad \sin^2 x + \cos^2 x = 1$

$= \sec^2(x)$

b. $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$

$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$

quotient rule
$f = \cos x \quad g = \sin x$
$f' = -\sin x \quad g' = \cos x$

$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$= \frac{-1(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= \frac{-1}{\sin^2 x}$ $\quad \sin^2 x + \cos^2 x = 1$

$= -\csc^2 x$

a. $\dfrac{d}{dx}(\tan x) = \sec^2 x$ **c.** $\dfrac{d}{dx}(\sec x) = \sec x \tan x$

b. $\dfrac{d}{dx}(\cot x) = -\csc^2 x$ **d.** $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

c. $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$

quotient rule $\frac{f'g - g'f}{g^2}$

$f = 1 \qquad g = \cos x$

$f' = 0 \qquad g' = -\sin x$

$= \frac{(0)(\cos x) - (-\sin x)(1)}{(\cos x)^2}$

$= \frac{\sin x}{\cos^2 x}$

$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$

$= \sec x \cdot \tan x$

d. $\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$

quotient rule $\frac{f'g - g'f}{g^2}$

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$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$

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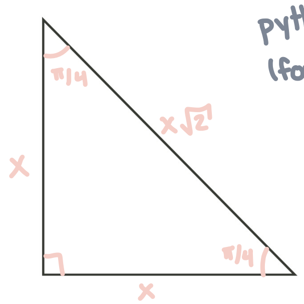


pythagorean theorem (for right triangles only):

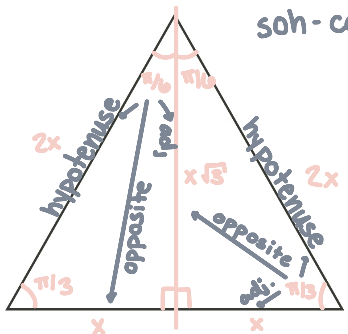
$a^2 + b^2 = c^2$

$x^2 + x^2 = c^2$

$2x^2 = c^2$

$\sqrt{2x^2} = \sqrt{c^2}$

$x\sqrt{2} = c$



soh-cah-toa : o = opposite

a = adjacent

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$$s(t) = \cos(t) + \sin(t) \quad \text{cm}$$

where t is in seconds.

(a) Find formulas for both its velocity and acceleration at time t seconds.

(b) Find smallest time at which the velocity of the piece of wood is zero.

(a) velocity = derivative of position; acceleration = derivative of velocity

$s(t) = \cos(t) + \sin(t)$

$v(t) = s'(t) = -\sin(t) + \cos(t)$

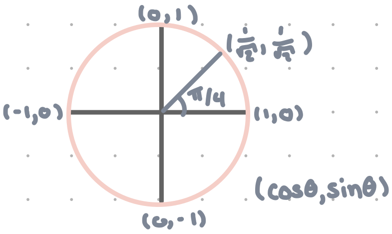
$a(t) = v'(t) = s''(t) = -\cos(t) - \sin(t)$

b. velocity is zero $\Rightarrow s'(t)=0$

$s'(t) = -\sin t + \cos t = 0$

$\cos t = \sin(t)$

$t = \pi/4$



5. Assuming that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$, answer the questions below:

a. Find the values of (i) $\lim_{x\to 0} \frac{\sin 7x}{3x}$ and (ii) $\lim_{x\to 0} \frac{\tan x}{2x}$

b. Show that the derivative of $\sin x$ is $\cos x$. You will need the identity $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

(a) i. $\lim_{x\to 0} \frac{\sin 7x}{3x} = \frac{1}{3} \lim_{x\to 0} \frac{\sin(7x)}{x} \cdot \frac{7}{7}$

$= \frac{7}{3} \lim_{x\to 0} \frac{\sin(7x)}{7x}$

let $u = 7x$

$= \frac{7}{3} \lim_{u\to 0} \frac{\sin(u)}{u}$

$= \frac{7}{3} \cdot 1$

ii. $\lim_{x\to 0} \frac{\tan x}{2x} = \lim_{x\to 0} \frac{\sin x}{2x \cos x}$

$= \lim_{x\to 0} \frac{\sin x}{2x} \cdot \frac{1}{\cos x}$

$= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$ $\quad \cos(0) = 1$

$= \frac{1}{2} \cdot 1 \cdot 1$

$= \frac{1}{2}$

(b) $\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$ $\qquad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$

$= \lim_{h\to 0} \frac{-\sin(x) + \sin(x)\cos(h) + \cos(x)\sin(h)}{h}$

$= \lim_{h\to 0} \frac{-\sin(x)[1-\cos(h)] + \cos(x)\sin(h)}{h}$

$= \lim_{h\to 0} \frac{-\sin(x)[1-\cos(h)]}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h}$

$= -\sin(x) \cdot \lim_{h\to 0} \frac{[1-\cos(h)]}{h} + \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}$

$= -\sin(x) \cdot 0 + \cos(x) \cdot 1$

$= \cos(x)$