Math 10350 – Example Set 06A
Section 3.7 The Chain Rule

Definition 1. (The Composite Function) A function $h(x)$ is said to be a composite function of $g(x)$ followed by $f(x)$ if $h(x) = f(g(x))$. We may write: $h : x \xmapsto{g} \underline{g(x)} \xmapsto{f} \underline{f(g(x))}$

1. Find functions $f(x)$ and $g(x)$, not equal x, such that $h(x) = f(g(x))$:

(a) $h(x) = (x^4 + 2x^2 + 7)^{21}$ $\qquad h : x \xmapsto{g} \underline{x^4+2x^2+7} \xmapsto{f} \underline{(x^4+2x^2+7)^{21}}$

Ans: $f(x) \overset{?}{=} \underline{x^4+2x^2+7}$ and $g(x) \overset{?}{=} \underline{x^2}$

(b) $h(x) = \sin(x^2 + 1)$ $\qquad h : x \longmapsto \underline{x^2+1} \longmapsto \underline{\sin(x^2+1)}$

Ans: $f(x) \overset{?}{=} \underline{x^2+1}$ and $g(x) \overset{?}{=} \underline{\sin(x)}$

World Guinness Record Approved Text: "The razor-toothed piranhas of the genera Serrasalmus and Pygocentrus are the most ferocious freshwater fish in the world. In reality they seldom attack a human."

Think about it: In a competition for the title of "Fastest Text Messager", it is observed that Competitor A inputs text three times faster than B, and Competitor B inputs text two times faster than C. How much faster is Competitor A than Competitor C? Why?

A — — $\overset{3}{\text{Rate of } A \text{ relative to } B}$ — — B — — $\overset{2}{\text{Rate of } B \text{ relative to } C}$ — — C

— — — — — — — — — — $\overset{6}{?\ \text{Rate of } A \text{ relative to } C\ ?}$ — — — — — — — — —

$A = 3 \cdot B$
$B = 2 \cdot C$ $\;>\; \underset{f}{3}\underset{g}{(2 \cdot C)} = 6C$

The Chain Rule. Suppose $y = f(g(x))$. To find a formula for $\dfrac{dy}{dx} = \dfrac{d}{dx}[f(g(x))]$, we set $u = g(x)$ then $y = f(u)$.

y — — $\overset{\frac{dy}{du}}{\text{Rate of } y \text{ relative to } u}$ — — u — — $\overset{\frac{du}{dx}}{\text{Rate of } u \text{ relative to } x}$ — — x

— — — — — — — — — $\overset{\frac{dy}{dx} \overset{?}{=} \frac{dy}{du} \cdot \frac{du}{dx}}{?\ \text{Rate of } y \text{ relative to } x\ ?}$ — — — — — — — — — —

So we expect $\boxed{\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}}$

Our guess is in fact correct, and the formula for $\dfrac{dy}{dx}$ is called the **Chain Rule** (in Leibniz notation).

But $\dfrac{dy}{dx} = \dfrac{d}{dx}[f(g(x))] = [f(g(x))]'$, $\dfrac{dy}{du} = f'(u) = f'(g(x))$ and $\dfrac{du}{dx} = g'(x)$. Thus we also have:

$$\boxed{\frac{d}{dx}[f(g(x))] = [f(g(x))]' = f'(g(x)) \cdot g'(x)}$$

outside derivative × inside derivative

1. Find the coordinates of the points on the curve $y = 2xe^{-x^2}$ where the tangent lines are horizontal.

2. Find the derivative of the following functions:

a. $f(x) = \sqrt[3]{1 + x^3}$

b. $g(x) = \cot 5x$

c. $h(x) = \dfrac{1}{2}x^2\sqrt{16 - x^2}$

d. $R = \csc^3(\pi x)$

e. $w = \left(\dfrac{t+1}{t^2+2}\right)^4$

f. $y = \tan^3(2x^2 + 1)$

1. Find the coordinates of the points on the curve $y = 2xe^{-x^2}$ where the tangent lines are horizontal.

horizontal tangent line => derivative is zero

$f(x) = \underline{2x}\,\underline{e^{-x^2}}$ product rule: $f'g + g'f$

$f'(x) = (2)(e^{-x^2}) + (-2xe^{-x^2})\cdot(2x)$

$= 2e^{-x^2} - 4x^2e^{-x^2}$

$= 2e^{-x^2}(1-2x^2)$

$y = e^{-x^2}$

$x \mapsto -x^2 \mapsto e^{-x^2}$

$f(x) = e^x$ $g(x) = -x^2$

$f'(x) = e^x$ $g'(x) = -2x$

$f'(g(x)) = e^{-x^2}$

$f'(g(x))\cdot g'(x) = e^{-x^2}\cdot(-2x)$

When is $f'(x) = 0$?

$f'(x) = 2e^{-x^2}(1-2x^2) = 0$ $a\cdot b = 0 \Rightarrow a = 0$ or $b = 0$

$2e^{-x^2} = 0$ $(1-2x^2) = 0$

$e^{-x^2} = 0$ $1 = 2x^2$

never $\frac{1}{2} = x^2$

$\pm\sqrt{\frac{1}{2}} = x$

2. Find the derivative of the following functions:

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c. $h(x) = \dfrac{1}{2}x^2\sqrt{16-x^2}$

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f. $y = \tan^3(2x^2+1)$

(a) $f(x) = \sqrt[3]{1+x^3}$ $f(u) = u^{1/3}$

$= (1+x^3)^{1/3}$ $f'(u) = \frac{1}{3}u^{1/3}$

$f'(x) = \frac{1}{3}(1+x^3)^{-2/3}\cdot 3x$

$= x(1+x^3)^{-2/3}$

(b) $g(x) = \cot(5x)$ $f(u) = \cot(u)$

$g'(x) = -\csc^2(5x)\cdot 5$ $f'(u) = -\csc^2(x)$

$= -5\csc^2(5x)$

(c) $h(x) = \frac{1}{2}x^2\sqrt{16-x^2}$ $g(u) = (16-x^2)^{1/2}$

$h(x) = \underbrace{\tfrac{1}{2}x^2}_{f}\,\underbrace{(\overset{u}{16-x^2})^{1/2}}_{g}$ $g'(u) = \frac{1}{2}(16-x^2)^{-1/2}\cdot -2x$

$h'(x) = f'g + [g(u)\cdot u']\cdot f$

$h'(x) = (x)(16-x^2)^{1/2} + [\,x(16-x^2)^{-1/2}\,]\cdot(\tfrac{1}{2}x^2)$

$= x\sqrt{16-x^2} + \frac{1}{2}x^3(16-x^2)^{-1/2}$

(d) $R = \csc^3(\pi x)$ this has two layers

$= (\csc(\pi x))^3$ $x \to \pi x \to \csc(\pi x) \to (\csc(\pi x))^3$

$R = f(u(v(x)))$ we have to peel back

(i) $f(u) = u^3 \Rightarrow f'(u) = 3u^2\cdot u'$

(ii) $u(v) = \csc(v) \Rightarrow u'(v) = -\csc(v)\cot(v)\cdot v'$

(iii) $v(x) = \pi x \Rightarrow v'(x) = \pi$

$R' = f'(u)\cdot u' = f'(u)\cdot[\,u'(v)\cdot v'\,]$

$= (3u^2)(-\csc(v)\cot(v))\cdot(\pi)$ sub in

$= 3[\csc(v)]^2(-\csc(v)\cot(v))(\pi)$ $u = \csc(v)$

$= 3[\csc(\pi x)]^2(-\csc(\pi x)\cot(\pi x))(\pi)$ $v = \pi x$

$= -3\pi\csc^3(\pi x)\cot(\pi x)$

(e) $w = \left(\dfrac{t+1}{t^2+2}\right)^4$ $f(u) = u^4 \Rightarrow f'(u) = 4u^3$

$u = \dfrac{t+1}{t^2+2}$

$w' = f'(u)\cdot u'$ $u' = \dfrac{(1)(t^2+2)-(2t)(t+1)}{(t^2+2)^2}$

$= 4\left(\dfrac{t+1}{t^2+2}\right)^3\cdot\dfrac{(t^2+2)-(2t)(t+1)}{(t^2+2)^2}$

$= 4\left(\dfrac{t+1}{t^2+2}\right)^3\cdot\dfrac{t^2+2-2t^2-2t}{(t^2+2)^2}$

$= 4\dfrac{(t+1)^3}{(t^2+2)^3}\cdot\dfrac{-t^2-2t+2}{(t^2+2)^2}$ $-t^2-2t+2$
$-(t^2+2t-2)$
does not factor

$= \dfrac{-4(t+1)^3(t^2+2t-2)}{(t^2+2)^5}$

(f) $y = \tan^3(2x^2+1)$ $y = f(u(v(x)))$

$= [\tan(2x^2+1)]^3$ $x \to 2x^2+1 \to \tan(2x^2+1)$

$y' = f'(u(v(x)))\cdot u'(v(x))\cdot v'(x)$ $\to (\tan(2x^2+1))^3$

$y' = 3[\tan(2x^2+1)]^2\cdot\sec^2(2x^2+1)\cdot(4x)$

Peel back the layers method:

$x \to v(x) \to u(v(x)) \to f(u(v(x)))$

deriving "removes" a function

derive f: $f'(u(v(x)))$

* derive u: $f'(u(v(x)))\cdot u'(v(x))$
* derive v: $f'(u(v(x)))\cdot u'(v(x))\cdot v'(x)$
* derive x: $f'(u(v(x)))\cdot u'(v(x))\cdot v'(x)\cdot 1$