

Math 10350 – Example Set 06B
Section 3.7 The Chain Rule
Section 3.9 Derivative of the Natural Log

1. Consider the functions $f(x) = e^x$ and $g(x) = \ln x$.

a. Sketch the graph of $f(x) = e^x$ and $g(x) = \ln x$ on the same set of axes. What could you say about their relationship? How are $f(x)$ and $g(x)$ related?

b. Using the fact that $\frac{d}{dx}(e^x) = e^x$ and the chain rule, find a formula for $\frac{d}{dx}(\ln x)$.

c. Using the change of base formula $\log_b x = \frac{\ln x}{\ln b}$, show that $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$.

2. Find the equation of the tangent line to the graph of $y = \frac{\ln x - 1}{\ln x + 1}$ when $x = 1$.

3. Find the derivatives of the following functions:

a. $f(\theta) = \ln(\sin \theta + 2)$

b. $y = \ln\left(\frac{e^x - 1}{e^x + 1}\right)$

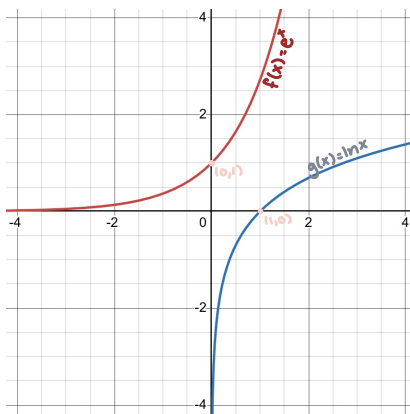
c. $g(z) = \ln(\ln z)$ for $z > 1$.

d. $y = e^{(\ln x)^3}$

e. $x^e + e^x$

e. x^x

1.(a) sketch $f(x) = e^x$, $g(x) = \ln x$



$f(g(x)) = e^{\ln(x)} = x$

$g(f(x)) = \ln(e^x) = x$

(b) Use $\frac{d}{dx}(e^x) = e^x$ & chain rule to find $\frac{d}{dx}(\ln x)$.

$\frac{d}{dx}(e^{\ln(x)}) = e^{\ln(x)} \cdot [\ln x]'$

"
 $\frac{d}{dx}(x) = 1$

thus $e^{\ln(x)} \cdot \frac{d}{dx}(\ln(x)) = 1$

$x \cdot \frac{d}{dx}(\ln(x)) = 1$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

similarly $\frac{d}{dx}(a^x) = a^x \ln(a)$
find $\frac{d}{dx}(\log_a(x))$

$\frac{d}{dx}(a^{\log_a(x)}) = a^{\log_a(x)} \cdot \ln(a) \cdot [\log_a(x)]'$

"
 $\frac{d}{dx}(x) = 1$

thus $a^{\log_a(x)} \cdot \ln(a) \cdot \frac{d}{dx}[\log_a(x)] = 1$

$x \cdot \ln(a) \cdot \frac{d}{dx}[\log_a(x)] = 1$

$\frac{d}{dx}[\log_a(x)] = \frac{1}{x \ln(a)}$

(c) use $\log_b x = \frac{\ln x}{\ln b}$ to find $\frac{d}{dx}(\log_b x)$

$\frac{d}{dx}[\log_b x] = \frac{d}{dx}\left[\frac{\ln(x)}{\ln(b)}\right] = \frac{1}{\ln(b)} \cdot \frac{d}{dx}[\ln x] = \frac{1}{\ln(b)} \cdot \frac{1}{x} = \frac{1}{x \ln(b)}$

↑
 $\ln(b)$ is just a constant

2. Find the equation of the tangent line to the graph of $y = \frac{\ln x - 1}{\ln x + 1}$ when $x = 1$.

$$y = \frac{\ln x - 1}{\ln x + 1} \quad \text{quotient rule: } \frac{f'g - g'f}{g^2}$$

$$y' = \frac{(\frac{1}{x}) \cdot (\ln x + 1) - (\frac{1}{x})(\ln x - 1)}{(\ln x + 1)^2} = \frac{\frac{1}{x} [(\ln x + 1) - (\ln x - 1)]}{(\ln x + 1)^2} = \frac{\frac{1}{x} [\cancel{\ln x + 1} - \cancel{\ln x} + 1]}{(\ln x + 1)^2} = \frac{\frac{1}{x} \cdot (2)}{(\ln x + 1)^2}$$

$$f'(1) = \frac{\frac{1}{1} \cdot 2}{(\ln(1) + 1)^2} = \frac{2}{(0+1)^2} = \frac{2}{1} = 2$$

$$f(1) = \frac{\ln(1) - 1}{\ln(1) + 1} = \frac{0 - 1}{0 + 1} = -1$$

tangent line: $y - f(x_1) = f'(x_1)(x - x_1)$

$$y - (-1) = (2)(x - 1)$$

$$y + 1 = 2x - 2$$

$$y = 2x - 3$$

3. Find the derivatives of the following functions:

a. $f(\theta) = \ln(\sin \theta + 2)$

b. $y = \ln\left(\frac{e^x - 1}{e^x + 1}\right)$

c. $g(z) = \ln(\ln z)$ for $z > 1$.

(a) $f(\theta) = \ln(\underbrace{\sin \theta + 2}_u)$ $f'(u) \cdot u'$

$$f'(\theta) = \frac{1}{\sin \theta + 2} \cdot \cos \theta$$

(b) $y = \ln\left(\frac{e^x - 1}{e^x + 1}\right)$ $h'(u) \cdot u'$ $u = \frac{e^x - 1}{e^x + 1}$

$$y' = h'(u) \cdot \frac{f'g - g'f}{g^2}$$

$$y' = \frac{1}{\frac{e^x - 1}{e^x + 1}} \cdot \left(\frac{(e^x)(e^x + 1) - (e^x)(e^x - 1)}{(e^x + 1)^2} \right)$$

$$= \frac{1}{1} \cdot \frac{e^x + 1}{e^x - 1} \cdot \left(\frac{\cancel{e^x} + e^x - \cancel{e^x} + e^x}{(e^x + 1)^2} \right)$$

$$= \frac{\cancel{e^x + 1}(2e^x)}{(e^x + 1)^2}$$

$$= \frac{2e^x}{e^x + 1}$$

(c) $g(z) = \ln(\underbrace{\ln(z)}_u)$

$$g'(z) = \frac{1}{\ln(z)} \cdot \frac{1}{z}$$

$$= \frac{1}{z \ln(z)}$$

d. $y = e^{(\ln x)^3}$

(d) $y = e^{(\ln x)^3}$

$$y' = e^{(\ln x)^3} \cdot \left[3(\ln x)^2 \cdot \frac{1}{x} \right]$$

e. $x^e + e^x$
 ↑ power ↑ exponential
 $e x^{e-1} + e^x$

e. x^x
 $e^{\ln(x^x)}$
 $e^{x \ln x}$

$$\begin{aligned} &e^{x \cdot \ln(x)} \cdot [1 \cdot \ln(x) + \frac{1}{x} \cdot x] \\ &e^{x \cdot \ln(x)} \cdot [\ln(x) + 1] \end{aligned}$$