Math 10350 – Example Set 07A
3.10 Related Rates

1. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/s. (a) How fast is the y-coordinate changing at this instant? (b) How fast is the distance from the particle to the origin changing at this instant? (Answer: (a) 3/4 cm/s; (b) $27/(4\sqrt{5})$ cm/s)

2. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is his shadow on the building decreasing when he is 4 m from the building. (Answer: 0.6 m/s)

3. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2? (Answer: $-8/5$ cm/min)

4. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s and remains taut, how fast is the angle of depression of the rope changing when the bow of the boat is 8 m from the dock? Note the angle of depression is the angle between the rope and the horizontal.

1. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/s. (a) How fast is the y-coordinate changing at this instant? (b) How fast is the distance from the particle to the origin changing at this instant? (Answer: (a) 3/4 cm/s; (b) $27/(4\sqrt{5})$ cm/s)

1. a particles position is described by the point $(x(t), y(t)) = (x(t), \sqrt{x(t)})$

(a) change in y-coordinate over time $\Rightarrow \frac{dy}{dt}$

careful to keep track of what you are taking the derivative with respect to $x'(t) = \frac{dx}{dt}$

$\frac{d}{dt}(y(t)) = \frac{d}{dt}(\sqrt{x(t)}) = \frac{d}{dt}((x(t))^{1/2}) = \frac{1}{2}(x(t))^{-1/2} \cdot x'(t) = \frac{1}{2\sqrt{x}} \cdot \frac{dx}{dt}$

$\frac{d}{dt}(y(t)) = \frac{1}{2\sqrt{x}} \cdot \frac{dx}{dt}$

given: $\frac{dx}{dt}(4,2) = 3$

at $(4,2)$: $\frac{1}{2\sqrt{4}} \cdot 3 = \frac{3}{4}$

(b) "rate of change" of distance from origin

distance: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

from $(0,0)$: $d_0 = \sqrt{(x_2)^2 + (y_2)^2}$

distance of $(x(t), \sqrt{x(t)})$: $d = \sqrt{(x(t))^2 + (\sqrt{x(t)})^2}$

$= ((x(t))^2 + (x(t)))^{1/2}$

$d' = \frac{1}{2}(x(t)^2 + x(t))^{-1/2} \cdot [2x(t) \cdot x'(t) + x'(t)]$

at $(4,2)$: $\frac{1}{2}((4)^2 + (4))^{-1/2} \cdot [2(4) \cdot 3 + 3]$

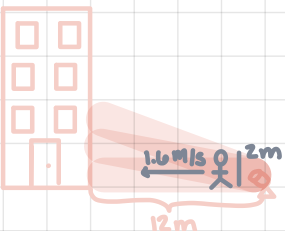
$= \frac{1}{2}(20)^{-1/2} \cdot [27]$

$= \frac{27}{2\sqrt{20}}$

$= \frac{27}{4\sqrt{5}}$

2. A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is his shadow on the building decreasing when he is 4 m from the building. (Answer: 0.6 m/s)

derivative should be negative
answer choice will be positive

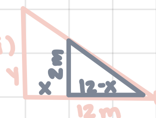


there are two ways to label this graph

(i) x is the distance between the man and the wall
↳ distance starts at 12 meters goes to zero; $\frac{dx}{dt} = -1.6$ m/s

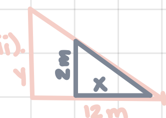
(ii) x is the distance between the man and the spotlight
↳ distance starts at zero and increase to 12; $\frac{dx}{dt} = 1.6$ m/s

(i)



triangle made by light
triangle made by man

(ii)



triangle made by light

similar triangles:

$\frac{\text{leg1 } \Delta 1}{\text{leg1 } \Delta 2} = \frac{\text{leg2 } \Delta 1}{\text{leg2 } \Delta 2}$

(i)

$\frac{y}{2} = \frac{12}{12-x}$

$y = \frac{24}{12-x}$

find $\frac{dy}{dt}$:

$\frac{d}{dt}\left[\frac{24}{12-x}\right] = \frac{d}{dt}[24(12-x)^{-1}]$

$= -24(12-x)^{-2} \cdot -1 \cdot \frac{dx}{dt}$

$= \frac{24}{(12-x)^2} \cdot (-1.6)$

at $x = 4$: $y'(4) = \frac{24}{(12-4)^2}(-1.6)$

$= -\frac{24}{64} \cdot 1.6$

$= -\frac{3}{8} \cdot \frac{8}{5}$

$= -\frac{3}{5} = -0.6$

(ii)

$\frac{y}{2} = \frac{12}{x}$

$y = \frac{24}{x}$

find $\frac{dy}{dt}$:

$\frac{d}{dt}\left[\frac{24}{x}\right] = \frac{d}{dt}[24x^{-1}]$

$= -24 \cdot x^{-2} \cdot \frac{dx}{dt}$

$= -\frac{24}{x^2} \cdot (1.6)$

4 meters from wall

at $x = 8$: $y'(8) = -\frac{24}{8^2}(1.6)$

$= -\frac{24}{64} \cdot 1.6$

$= -\frac{3}{8} \cdot \frac{8}{5}$

$= -\frac{3}{5} = -0.6$

height

3. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2? (Answer: $-8/5$ cm/min)

given: height changes over time $\frac{dh}{dt} = 1$
area changes over time $\frac{dA}{dt} = 2$

asked for: $\frac{db}{dt}$ at $h = 10$, $A = 100$

Area $= A(t) = \frac{1}{2} b(t) \cdot h(t)$

derive: $\frac{d}{dt}(A(t)) = \frac{1}{2}\left[\frac{d}{dt}(b(t) \cdot h(t))\right]$

$\frac{d}{dt}(A(t)) = \frac{1}{2}\left[\frac{d}{dt}(b(t))h(t) + \frac{d}{dt}(h(t))b(t)\right]$

$A'(t) = \frac{1}{2}[b'(t) \cdot h(t) + h'(t) \cdot b(t)]$

first: find $b(t)$ for $h = 10, A = 100$

$A = \frac{1}{2} bh$

$100 = \frac{1}{2} b \cdot 10$

$20 = b$

second: plug into $A'(t)$

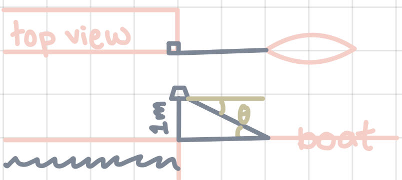
$2 = \frac{1}{2}[b'(t) \cdot 10 + 1 \cdot 20]$

$4 = 10 \cdot b'(t) + 20$

$-16 = 10 \cdot b'(t)$

$-\frac{8}{5} = b'(t)$

4. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s and remains taut, how fast is the angle of depression of the rope changing when the bow of the boat is 8 m from the dock? Note the angle of depression is the angle between the rope and the horizontal.



given: $y = 1$ (always), $\frac{dh}{dt} = -1$ m/s, $x =$ distance between boat and dock

asked for: change in θ i.e. $\frac{d\theta}{dt}$ (θ can be found two ways)

Step 1: find a relation between the variables

$\left.\begin{array}{l} \sin\theta = \frac{1}{h} \\ \cos\theta = \frac{x}{h} \\ \tan\theta = \frac{1}{x} \end{array}\right\}$ too many variables

Step 2: derive implicitly with respect to t

$\frac{d}{dt}[\sin\theta = h^{-1}]$: $\cos\theta \cdot \frac{d\theta}{dt} = -1 h^{-2} \cdot \frac{dh}{dt}$

$\cos\theta \frac{d\theta}{dt} = -\frac{1}{h^2} \cdot \frac{dh}{dt}$

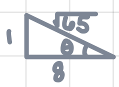
Step 3: solve for $\frac{d\theta}{dt}$

$\cos\theta \cdot \frac{d\theta}{dt} = -\frac{1}{h^2} \cdot \frac{dh}{dt}$

$\frac{d\theta}{dt} = -\frac{1}{\cos\theta h^2} \cdot \frac{d\theta}{dt}$

Step 4: plug in what you know for $x = 8$

(you secretly know θ. $h^2 = (1)^2 + (8)^2$ and $\cos\theta = \frac{\sqrt{65}}{8}$)



$\frac{d\theta}{dt} = -\frac{1}{(\sqrt{65}/8) \cdot 65} \cdot -1$

$= \frac{1/1}{(65\sqrt{65}/8)}$

$= \frac{1}{1} \cdot \frac{8}{65\sqrt{65}}$

$= \frac{8\sqrt{65}}{4225}$