

Math 10350 – Example Set 07C
Section 4.1 Linear Approximation and Applications

1. The population of wolves $w(t)$ and wild boars $p(t)$ in the thousands are given by the equations:

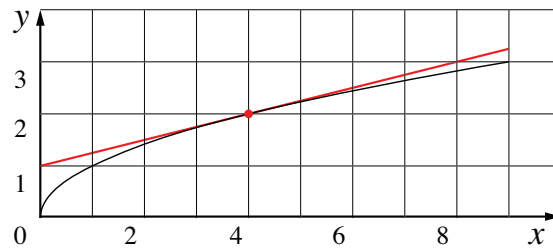
$$w(t) = 3 \sin t + 5; \quad p(t) = 2 \cos t + 5.$$

(a) What is the rate of change of w with respect to p at $t = \frac{\pi}{4}$? (b) Find a relation between w and p by eliminating t . (c) Draw the graph of the p and w relationship in a p - w coordinate plane, (d) Describe what is happening between the two populations as time t progresses. Hint: Input different values of t and trace the curve you drew.

1. Find the tangent line to $f(x) = \sqrt{x}$ at $x = 4$.

(b) Write down the linearization (linear approximation) of $f(x) = \sqrt{x}$ at $x = 4$.

linearization: $f(x) \approx f'(x_1)(x-x_1) + f(x_1)$
 $f'(x) = \frac{1}{2} x^{-1/2}$ $f(x) = \frac{1}{4}(x-4) + 2$
 $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ $= \frac{1}{4}x - 1 + 2$
 $f(4) = \sqrt{4} = 2$ $= \frac{1}{4}x + 1$



(c) Using your answer in (b), estimate the following values and comment on their accuracy with a calculator:

(i) $f(4.05) \approx 2.0125$ (ii) $f(3.9) \approx 1.975$ (iii) $f(5) \approx 2.25$

2. Find the linearization (tangent line approximation) of e^x at $x = 0$. Estimate $e^{0.04}$. Draw a graph to illustrate your estimation. Is your estimate an overestimate or underestimate?

Linear Approximation of change in a function. The linearization of $f(x)$ at $x = a$ is often used in estimating the change Δf of a function $f(x)$ as x changes from a to $a + \Delta x$ is often difficult to compute exactly. Draw in the graph below to show where Δf is.

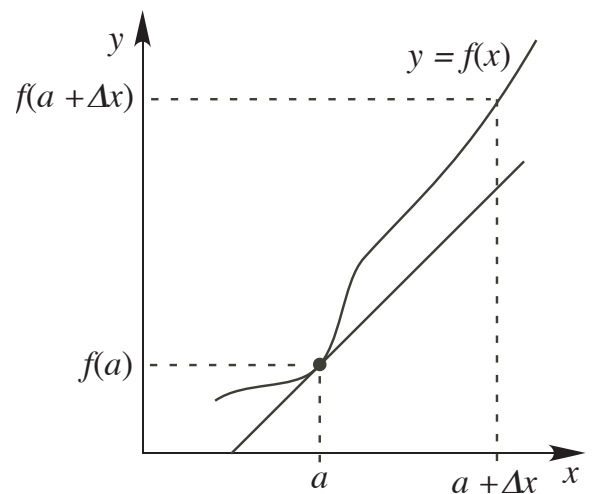
(a) Exact value of $\Delta f = f'(a)(x-a)$

(b) For small Δx , the linear approximation of $f(x)$ at $x = a$ gives:

$$\Delta f \approx f'(a)\Delta x$$

(c) Such estimates for Δf are often used to approximate change and percentage change.

percent change: $100 \cdot \frac{\Delta f}{f}$



3. (Concept Test) If $g(3) = 4$ and $g'(3) = -1$. Estimate Δg and the percentage change of g as x changes from 3 to 3.01. Estimate $g(3.01)$.

1. The population of wolves $w(t)$ and wild boars $p(t)$ in the thousands are given by the equations:

$$w(t) = 3 \sin t + 5; \quad p(t) = 2 \cos t + 5.$$

(a) What is the rate of change of w with respect to p at $t = \frac{\pi}{4}$? (b) Find a relation between w and p by eliminating t . (c) Draw the graph of the p and w relationship in a p - w coordinate plane, (d) Describe what is happening between the two populations as time t progresses. Hint: Input different values of t and trace the curve you drew.

(a) find $\frac{dw}{dp}$

$$\frac{dw}{dp} \cdot \frac{dp}{dt} = \frac{dw}{dt}$$

$$\frac{dw}{dp} = \left(\frac{dw}{dt}\right) / \left(\frac{dp}{dt}\right)$$

$$\frac{dw}{dt} = 3 \cos(t) + 0$$

$$\frac{dp}{dt} = -2 \sin(t) + 0$$

$$\frac{dw}{dp} = \frac{3 \cos(t)}{-2 \sin(t)}$$

$$\frac{dw}{dp} = -\frac{3}{2} \cot(t)$$

$$\frac{dw}{dp}\left(\frac{\pi}{4}\right) = -\frac{3}{2} \cot\left(\frac{\pi}{4}\right)$$

$$= -\frac{3}{2}(1)$$

$$= -\frac{3}{2}$$

(b) $w = 3 \sin t + 5 \rightarrow t = \arcsin\left(\frac{w-5}{3}\right)$

$$p = 2 \cos t + 5$$

$$p = 2 \cos\left(\arcsin\left(\frac{w-5}{3}\right)\right) + 5$$

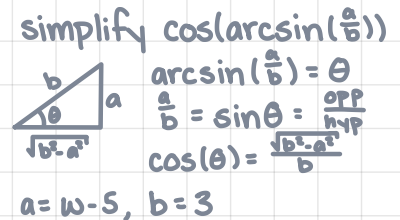
$$p = 2 \left(\frac{\sqrt{9 - (w-5)^2}}{3}\right) + 5$$

$$p = \frac{2}{3} \sqrt{9 - w^2 + 10w - 25} + 5$$

$$= \frac{2}{3} \sqrt{-w^2 + 10w - 16} + 5$$

$$= \frac{2}{3} \sqrt{-(w^2 - 10w + 16)} + 5$$

$$= \frac{2}{3} \sqrt{-(w-8)(w-2)} + 5$$



(b) alternative: utilize $\sin^2(t) + \cos^2(t) = 1$

$$w = 3 \sin t + 5 \Rightarrow \sin t = \frac{w-5}{3}$$

$$p = 2 \cos t + 5 \Rightarrow \cos t = \frac{p-5}{2}$$

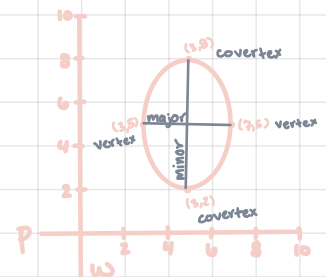
$$1 = (\sin t)^2 + (\cos t)^2 = \left(\frac{w-5}{3}\right)^2 + \left(\frac{p-5}{2}\right)^2$$

$$1 = \frac{1}{9}(w-5)^2 + \frac{1}{4}(p-5)^2$$

(c) graph $(x, y) \rightarrow (p, w)$

ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

- vertices: $(h \pm a, k)$
- length of major axis: $2a$
- covertices: $(h, k \pm b)$
- length of minor axis: $2b$



2. Find the linearization (tangent line approximation) of e^x at $x = 0$. Estimate $e^{0.04}$. Draw a graph to illustrate your estimation. Is your estimate an overestimate or underestimate?

linearization: $f(x) \approx f'(x_1)(x - x_1) + f(x_1)$

$$f'(x) = e^x \quad f(x) \approx 1(x-0) + 1$$

$$f'(0) = e^0 = 1 \quad = x + 1$$

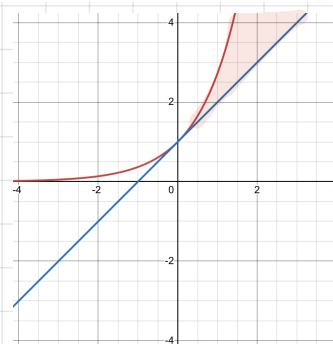
$$f(0) = e^0 = 1$$

$$f(0.04) \approx 0.04 + 1 = 1.04$$

$$f(0.04) = e^{0.04} = 1.0408$$

$$f(1) \approx 1 + 1 = 2$$

$$f(1) = e = 2.71$$



underestimate

3. (Concept Test) If $g(3) = 4$ and $g'(3) = -1$. Estimate Δg and the percentage change of g as x changes from 3 to 3.01. Estimate $g(3.01)$.

linearization: $f(x) \approx f'(x_1)(x - x_1) + f(x_1)$

$$\Delta g = g(x) - g(x_1) \approx g'(x_1)(x - x_1)$$

$$\Delta g \approx g'(3)(x - 3)$$

$$= -1(x - 3)$$

$$= -x + 3$$

$$g(x) \approx -1(x - 3) + 4$$

$$= -x + 3 + 4$$

$$= -x + 7$$

$$g(3.01) = -3.01 + 7$$

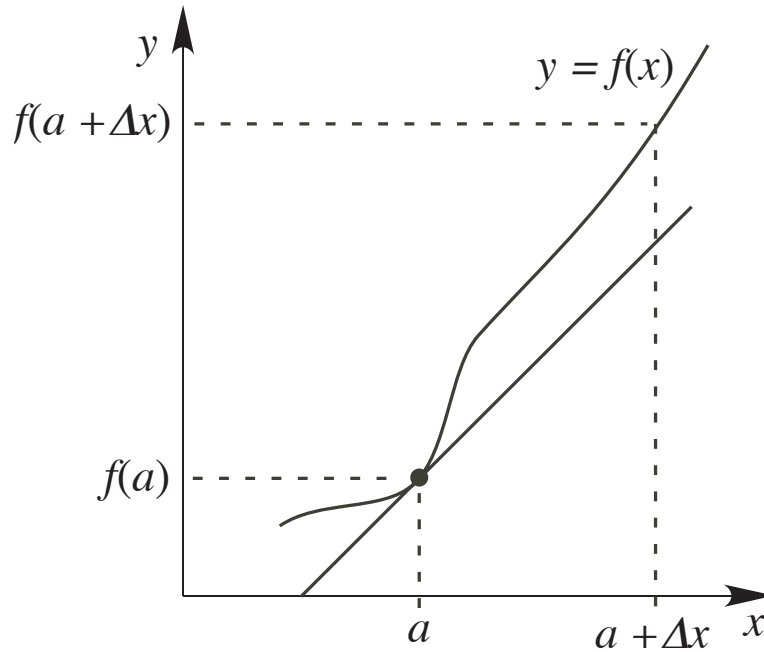
$$= 3.99$$

percent: $100 \cdot \frac{\Delta g}{g}$

$$100 \cdot \frac{-3.01 + 3}{4}$$

$$= 100 \cdot \frac{-0.01}{4} = -\frac{1}{4} = -25\%$$

Summary: Linearization of a Differentiable Function at $x = a$



The linear approximation (or linearization or tangent line approx.) of a differentiable function $f(x)$ at $x = a$ is given by the function of the _____ to the graph of $f(x)$ at $x = a$.

$$f(x) \approx L(x) = \underline{f'(a)(x-a) + f(a)}$$

this is just the equation of the tangent line at the point $(a, f(a))$ in a simplified format

(a) Exact value of $\Delta f = \underline{f(a + \Delta x) - f(a)}$

(b) For small Δx , the change in $f(x)$ as x changes from a to $a + \Delta x$ is given by:

$$\Delta f \approx f'(a) \Delta x$$

(c) Such estimates for Δf are often used to approximate change and percentage change.