Math 10350 – Optimization on Closed and Bounded Interval

The extreme value theorem

If $f(x)$ is continuous on a closed and bounded interval $[a, b]$ then $f(x)$ takes on

a maximum and takes on a minimum on $[a, b]$.

On a closed and bound interval $[a, b]$, a continuous function $f(x)$ attains its absolute maximum and absolute minimum occur at the following possible locations

(1) $x = a$ or

(2) $x = b$ or

(3) $a < x < b$.

Definition: Let $f(x)$ be defined at c. Then we say that c is a **critical point** of f if (A) $f'(x) = 0$,

or (B) $f'(x)$ DNE.

Method for finding absolute maxima and minima of f on $[a, b]$

1. Find all critical points in (a, b).
2. Evaluate f at all critical points and at endpoints. Then compare the values of f:

   highest = absolute maximum and **lowest** = absolute minimum.

Attendance

Q: Let $f(x) = \frac{2x-4}{\sqrt{6-3x}}$. Find $f'(x)$.

(a) $-\frac{9x-18}{(6-3x)^{1/2}}$

(b) $-\frac{2}{3\sqrt{6-3x}}$ wrong fraction

(c) $(6-3x)^{1/2}$

(d) $2\sqrt{6-3x} + \frac{3(2x-4)}{2(6-3x)^{1/2}}$

$f(x) = (2x-4)(6-3x)^{1/2}$

$f(x) = \frac{2x-4}{(6-3x)^{1/2}} = (2x-4)(6-3x)^{1/2}$

$f'(x) = \frac{2(6-3x)^{1/2} - \frac{1}{2}(6-3x)^{-1/2} \cdot (-3) \cdot (2x-4)}{((6-3x)^{1/2})^2} \cdot \frac{(6-3x)^{1/2}}{(6-3x)^{1/2}}$

$= \frac{2(6-3x) + 3(x-2)\cancel{(6-3x)^0}}{(6-3x)^{3/2}}$

$= \frac{12-6x+3x-6}{(6-3x)^{3/2}}$

$= \frac{6-3x}{(6-3x)^{3/2}} = (6-3x)^{1/2}$

Math 10350 – Example Set 10A

Section 4.2 Extreme Values

1. A piece of wire $10\,m$ long is cut into two pieces. One piece is bent into a square and the other is bent into a circle. How should the wire be cut so that the total area enclosed is (a) maximum and (b) minimum.

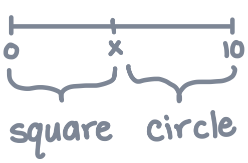
Section 4.3 The Mean Value theorem

> **The Mean Value theorem**
> If $f(x)$ is continuous on $[a,b]$ and differentiable on the open interval (a,b) then there exists a number c in (a,b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

2. Verify that the function $f(x) = \dfrac{x}{x+2}$ satisfies the hypotheses of the Mean Value Theorem on $[1,4]$. Then find all numbers c that satisfies the conclusion of the Mean Value Theorem.

3. A police observed Kelly Brian's car passing his check point 8:00am at 62 mph. His patrolling buddy further down the 65mph-limit highway observed Kelly Brian's car 8:45am at 65 mph. Assuming that the highway is fairly straight and the two police are about 56 miles apart, can you conclude that Kelly Brian has been obeying the speed limit in the duration between 8am to 8:45am?

1. $\ell = 10m$, $x =$ perimeter of square $= 4s$, $10-x =$ circumference of circle $= 2\pi r$



Area of square $= s^2$, Area of circle $= \pi r^2$, Total Area $= s^2 + \pi r^2$

we want to maximize total area with respect to our cut x, i.e. $\frac{dA}{dx}$

↳ we can rewrite area as a function of x or use implicit differentation

rewrite total area

$x = 4s \Rightarrow \frac{1}{4}x = s$

$10-x = 2\pi r \Rightarrow \frac{10-x}{2\pi} = r$

$A = s^2 + \pi r^2$
$= (\frac{1}{4}x)^2 + \pi(\frac{10-x}{2\pi})^2$
$= \frac{1}{16}x^2 + \frac{\pi}{4\pi^2}(10-x)^2$ ↙ you could multiply out
$= \frac{1}{16}x^2 + \frac{1}{4\pi}(10-x)^2$

find critical points

$A = \frac{1}{16}x^2 + \frac{1}{4\pi}(10-x)^2$

$\frac{dA}{dx} = \frac{2}{16}x + \frac{-2}{4\pi}(10-x)$

$0 = \frac{1}{8}x - \frac{1}{2\pi}(10-x)$

$0 = \frac{1}{8}x - \frac{5}{\pi} + \frac{x}{2\pi}$

$0 = \frac{2\pi x + 8x}{16\pi} - \frac{5}{\pi}$

$8\pi \cdot \frac{5}{\pi} = \frac{\pi x + 4x}{8\pi} \cdot 8\pi$

$40 = \pi x + 4x$

$40 = x(\pi + 4)$

$\frac{40}{\pi+4} = x$

test critical & end points

$A = \frac{1}{16}x^2 + \frac{1}{4\pi}(10-x)^2$

$A(\frac{40}{\pi+4}) = \frac{1}{16}(\frac{40}{\pi+4})^2 + \frac{1}{4\pi}(10 - \frac{40}{\pi+4})^2$
≈ 3.5006

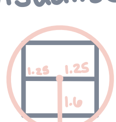
$A(0) = 0 + \frac{1}{4\pi}(10-0)^2$
$= \frac{100}{4\pi} = \frac{25}{\pi} \approx 7.9577$

$A(10) = \frac{1}{16}(10)^2 + \frac{1}{4\pi}(10-10)^2$
$= \frac{100}{16} + 0 = \frac{25}{4} \approx 6.25$

$s=0$, $r=\frac{5}{\pi}$ → maximum area when $x=10$

minimum area when $x = \frac{40}{\pi+4}$

visualize the answer:

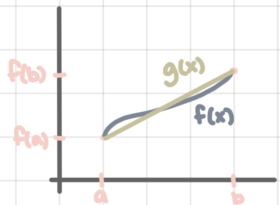


$s = 2.5$
$r = \frac{5}{\pi} = 1.6$

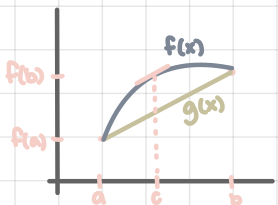
Section 4.3 The Mean Value theorem

> **The Mean Value theorem**
> If $f(x)$ is continuous on $[a,b]$ and differentiable on the open interval (a,b) then there exists a number c in (a,b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proof by Picture:



g(x)= secant line from f(a) to f(b)
point-slope form:
$g(x) - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$



2. Verify that the function $f(x) = \dfrac{x}{x+2}$ satisfies the hypotheses of the Mean Value Theorem on $[1,4]$. Then find all numbers c that satisfies the conclusion of the Mean Value Theorem.

(i) continuous on [1,4]
↳ only discontinuity at $x=-2$
(ii) differentiable on [1,4]
↳ only non-differentiable at $x=-2$

$f(x) = \frac{x+2}{x+2} - \frac{2}{x+2}$
$= 1-2(x+2)^{-1}$
$f'(x) = 0 + 2(x+2)^{-2}\cdot 1$
$= \frac{2}{(x+2)^2}$

$f(4) = \frac{4}{4+2}$
$= \frac{4}{6}$
$= \frac{2}{3}$
$f(1) = \frac{1}{1+2}$
$= \frac{1}{3}$

$\frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(1)}{4-1}$
$= \frac{2/3 - 1/3}{3}$
$= \frac{1/3}{3/1}$
$= \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$

$f'(x) = \frac{f(b)-f(a)}{b-a}$
$\frac{2}{(x+2)^2} = \frac{1}{9}$
$18 = (x+2)^2$
$18 = x^2+4x+4$
$0 = x^2+4x-14$

quadratic formula
$x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$
$= \frac{-4\pm\sqrt{4^2-4\cdot1\cdot(-14)}}{2\cdot 1}$
$= \frac{-4\pm\sqrt{16+56}}{2}$
$= \frac{-4\pm\sqrt{72}}{2}$
$= \frac{-4\pm 6\sqrt{2}}{2}$
$= -2\pm 3\sqrt{2}$

3. A police observed Kelly Brian's car passing his check point 8:00am at 62 mph. His patrolling buddy further down the 65mph-limit highway observed Kelly Brian's car 8:45am at 65 mph. Assuming that the highway is fairly straight and the two police are about 56 miles apart, can you conclude that Kelly Brian has been obeying the speed limit in the duration between 8am to 8:45am?

f(x)= Kelly Brian's position
f'(x)= Kelly Brian's velocity
↳|velocity|= speed

Use mean value theorem to see if $f'(x) > 65$
↳ $f(b)-f(a)$ = distance traveled = 56 mile
↳ $b-a$ = time passed = 45 minutes = $\frac{45}{60}$ hours (3/4)

$\frac{f(b)-f(a)}{b-a} = \frac{56}{3/4} = \frac{56}{1}\cdot\frac{4}{3} = \frac{224}{3} \approx 74.67$