Section 4.4: Second Derivative and Concavity

1. The graph of the **derivative** $f'(x)$ of $f(x)$ is given below. Find all critical points of $f(x)$ and use the derivative to determine where the function is increasing, where it is decreasing, and where it has a local maximum and minimum, if any.

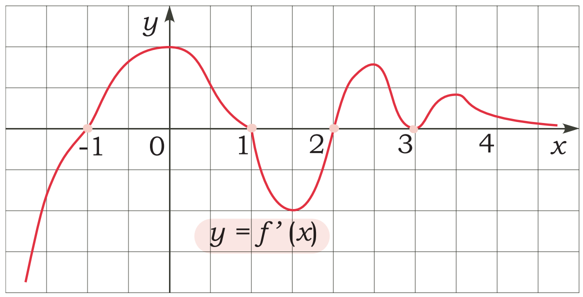
Please pay attention to labels. They have given $f'(x)$, but are asking about $f(x)$.

critical points: $f'(x)=0$ or DNE
$x=-1,1,2,3$

$f(x)$ is increasing: $f'(x)>0$
$(-1,1)\cup(2,3)\cup(3,\infty)$

careful: $f'(3)=0$ not positive

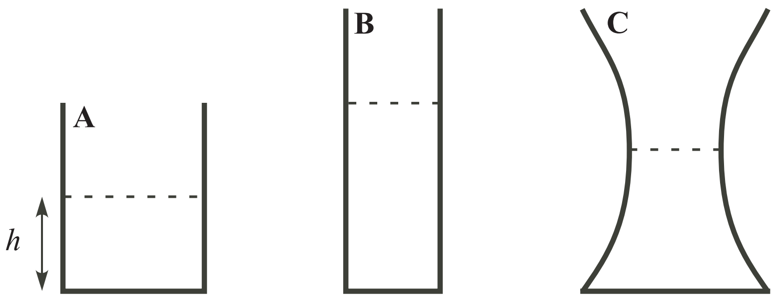
$f(x)$ is decreasing: $f'(x)<0$
$(-\infty,-1)\cup(1,2)$



maximum: inc. to dec.
pos. to neg.
$x=1$

minimum: dec. to inc
neg. to pos.
$x=-1,2$

2. Water is filling up each of the following vessels at a constant rate of 1 cm^3/sec.



Let h be the height of the water level in the vessel at time t.

a. Sketch the graphs of h verses t for Vessels A and B in the axes for Figure 1. Indicate which graph belong to A and which to B.

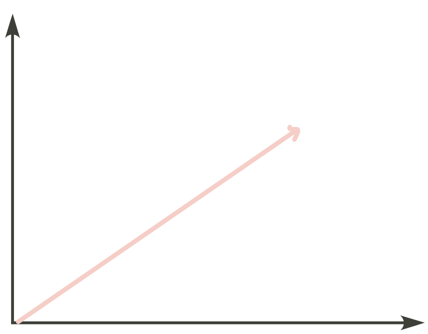


Figure 1

for A

Given $\frac{dV}{dt} = 1\,cm^3/sec$

Asked for graph of $h(t)$

reminder: $V = bh$
so when the base is bigger the height will grow slower

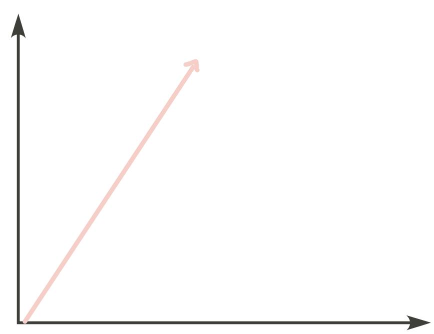
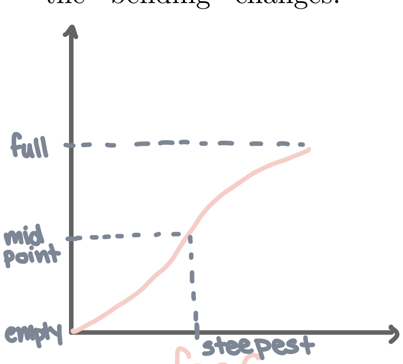


Figure 2

for B

b. Sketch the graph of h verses time t for Vessel C in the axes for Figure 2.

c. Comment on how the "bending" (up or down) of the graph changes with $h'(t)$. Mark on the graph where the "bending" changes.



for C

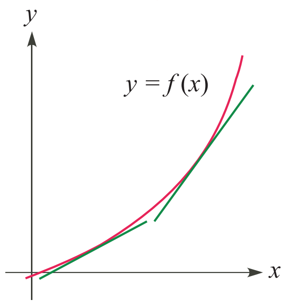
at the bottom C is bigger so h will increase slower then it will speed up until the midpoint (read as narrowest) after which it will increase slower again

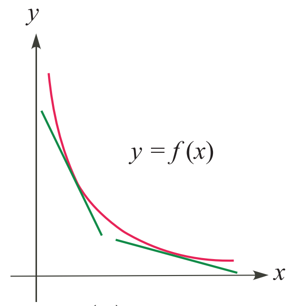
bend up meant it was filling up faster i.e. the speed of which it was filling up was increasing (second derivative)

so bending up (concave up) means $f''(x)>0$ and down (concave down) means $f''(x)<0$

Characterization of Concavity

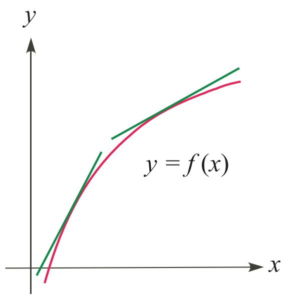
Case 1: For $a < x < b$, slope of the graph $f(x)$ is **increasing** as x increases i.e. $f'(x)$ is increasing. So $f''(x)$ is bent up for $a < x < b$. (Portions of u-shape)

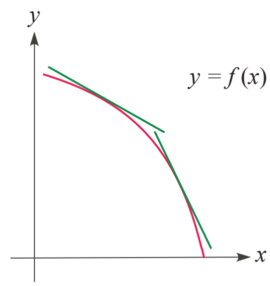




We say that the graph of $f(x)$ is concave up for $a < x < b$.

Case 2: For $a < x < b$, slope of the graph $f(x)$ is **decreasing** as x increases i.e. $f'(x)$ is decreasing. So $f''(x)$ is bent down for $a < x < b$. (Portions of n-shape)





We say that the graph of $f(x)$ is concave down for $a < x < b$.

Definition (Inflection Points or Points of Inflection) We say that $(c, f(c))$ is a point of inflection of $f(x)$ if $f'(c)$ exist (so graph has tangent line at $x = c$), and the graph of $f(x)$ changes concavity at $x = c$.

Remark: Graph changes concavity at inflection point so to locate points of inflection we need to check the signum of the second derivative near these places in the domain:

(1) $f''(x) = 0$, (2) $f''(x)$ DNE

Note that the graph could also change its concavity at vertical asymptotes where neither $f(c)$, $f'(c)$ and $f''(c)$.
exists

3. Find all points of inflection and concavity of $f(x) = \frac{9}{4}x^{4/3} - \frac{1}{6}x^3 + 3$.

While $f''(0)$ DNE, it is not an inflection point.
Inflection point = change in concavity.

Second Derivative Test
Let $f(x)$ be a function such that $f'(c) = 0$ and the function has a second derivative in an interval containing c.

- If $\boldsymbol{f''(c) > 0}$ then f has minimum at the point $(c, f(c))$.
- If $\boldsymbol{f''(c) < 0}$ then f has maximum at the point $(c, f(c))$.
- If $\boldsymbol{f''(c) = 0}$ then inconclusive. Use first derivative test.

4. Find all relative extrema for the function $f(x) = \frac{9}{4}x^{4/3} - \frac{1}{6}x^3 + 3$. Use second derivative test to classify them whenever applicable. What would you do when the second derivative test does not apply at a critical point?

3. Find all points of inflection and concavity of $f(x) = \frac{9}{4}x^{4/3} - \frac{1}{6}x^3 + 3$.

$f(x) = \frac{9}{4}x^{4/3} - \frac{1}{6}x^3 + 3$

$f'(x) = \frac{9}{4}\cdot\frac{4}{3}x^{1/3} - \frac{1}{6}\cdot 3x^2 + 0$

$= 3x^{1/3} - \frac{1}{2}x^2$

$f''(x) = 3\cdot\frac{1}{3}x^{-2/3} - \frac{1}{2}\cdot 2x^1$

$= x^{-2/3} - x$

$0 = \frac{1}{\sqrt[3]{x^2}} - x \longrightarrow f(0)$ DNE

$x^{2/3}\cdot x = \frac{1}{x^{2/3}}\cdot x^{2/3}$

$x^{5/3} = 1$

$x = 1$

POSSIBLE points of inflections: $f''(x) = 0$ and $f''(x)$ DNE

concave up: $f''(x) > 0$

concave down: $f''(x) < 0$

point(s) of inflection: $x = 0, 1$

\+ + −

0 1

test $x=-1$ test $x=\frac{1}{8}$ test $x=8$

$f''(-1) = \frac{1}{\sqrt[3]{(-1)^2}} - (-1)$

$= 1 + 1 = 2$

positive

$f''(\frac{1}{8}) = \frac{1}{\sqrt[3]{(1/8)^2}} - (\frac{1}{8})$

$= \frac{1}{\sqrt[3]{1/64}} - \frac{1}{8}$

$= \frac{1}{1/4} - \frac{1}{8} = 4 - \frac{1}{8}$

positive

$f(8) = \frac{1}{\sqrt[3]{(8)^2}} - (8)$

$= \frac{1}{\sqrt[3]{64}} - 8$

$= \frac{1}{4} - 8$

negative

concave up: $(-\infty, 0) \cup (0, 1)$

concave down: $(1, \infty)$

4. Find all relative extrema for the function $f(x) = \frac{9}{4}x^{4/3} - \frac{1}{6}x^3 + 3$. Use second derivative test to classify them whenever applicable. What would you do when the second derivative test does not apply at a critical point?

$f'(x) = 3\sqrt[3]{x^1} - \frac{1}{2}x^2 = 0$

$(3\sqrt[3]{x^1} = \frac{1}{2}x^2)^3$

$27x = \frac{1}{8}x^6$

$27x - \frac{1}{8}x^6 = 0$

$x(27 - \frac{1}{8}x^5) = 0$

$x = 0$

$27 - \frac{1}{8}x^5 = 0$

$27 = \frac{1}{8}x^5$

$216 = x^5$

$\sqrt[5]{216} = x$

$f''(x) = \frac{1}{\sqrt[3]{x^2}} - x$

$f''(0) =$ DNE

$f''(216^{1/5}) = ((216)^{1/5})^{-2/3} - (216)^{1/5}$

$= \underbrace{(216)^{-2/15}}_{<1} - \underbrace{(216)^{1/5}}_{>1}$

negative

216

4 54

2 2 2 27

3 9

3 3

$x=0$: $f'(x) = 0$ and $f''(x) =$ DNE so second derivative test is inconclusive

$x = (216)^{1/5}$: $f'(x) = 0$ and $f''(x) < 0$ so is a maximum

Since the second derivative test is inconclusive at $x=0$, we must use the first derivative test so we build a y-prime sign line.

$f'(x) = 3\sqrt[3]{x^1} - \frac{1}{2}x^2$

− + max

0 $(216)^{1/5}$

test $x=-1$ test $x=1$

$f'(-1) = 3\sqrt[3]{(-1)^1} - \frac{1}{2}(-1)^2$

$= 3(-1) - \frac{1}{2}(1)$

$= -3 - \frac{1}{2}$

negative

$f(1) = 3\sqrt[3]{(1)^1} - \frac{1}{2}(1)^2$

$= 3(1) - \frac{1}{2}(1)$

$= 3 - \frac{1}{2}$

positive

minimum at $x = 0$