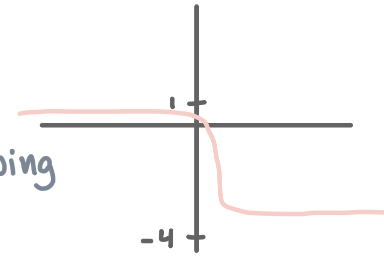
Math 10350 – Example Set 11A

Vertical Asymptote.
Let c be a real number. If $\lim_{x\to c^-} f(x) = \pm\infty$ or $\lim_{x\to c^+} f(x) = \pm\infty$.

Then $y = f(x)$ has a vertical asymptote at $x = c$.

Horizontal Asymptote.
If $\lim_{x\to\infty} f(x) = A$ (finite number) or $\lim_{x\to-\infty} f(x) = A$.

Then $y = f(x)$ has a horizontal asymptote at $y = A$.



1. Draw a graph with horizontal asymptotes $y = 1$ and $y = -4$.
aka "end behaviors"
↳ what does the graph do at the end of your drawing

2. Find the equations of all horizontal asymptotes of $y = \dfrac{3e^{3x} + 4e^x + 5}{2e^{3x} + e^x + 3}$.

L'Hopital's Rule: If both $f(x)$ and $g(x)$ are differentiable functions such that:

(a) $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$ such that $\lim_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \lim_{x\to c} \dfrac{f'(x)}{g'(x)}$. i.e. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}$

(b) $\lim_{x\to c} f(x) = \pm\infty = \lim_{x\to c} g(x)$ such that $\lim_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \lim_{x\to c} \dfrac{f'(x)}{g'(x)}$. i.e. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

Here $x \to c$ could mean limit to a number like $x \to 4$, or left-right limit notations like $x \to 0^-$ and $x \to 0^+$, or limit to infinity ($x \to \infty$ and $x \to -\infty$).

3. Evaluate the following limits using L'Hopital's Rule where necessary.

(A) 0/0 - type, ∞/∞ - type and $0 \cdot \infty$ - type

(i) $\lim_{x\to\infty} \dfrac{\ln(1+x)}{x}$.

(ii) $\lim_{x\to\infty} \dfrac{\sin(x) + \sin(2x)}{x^2 + 1}$.

(iii) $\lim_{x\to 0^+} x\ln(x)$.

(B) 1^∞ - type, ∞^0 - type and 0^0 - type

(iv) $\lim_{x\to\infty} (1+x)^{1/x}$.

(v) $\lim_{x\to\infty} \left(1 - \dfrac{2}{x}\right)^x$.

(vi) $\lim_{x\to 0^+} x^x$.

Key takeaways:
- if $\lim_{x\to} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ or $\frac{0}{0}$ we can apply L'H
- if $\lim_{x\to} f(x)^{g(x)} = 0^0$ or 1^∞ or ∞^0 we can use $e^{\ln(f(x))} = f(x)$ to make the problem: $e^{\lim_{x\to} g(x)\cdot\ln(f(x))} = e^{0\cdot\infty}$ or $e^{\infty\cdot 0}$
- if $\lim_{x\to 0} f(x)\cdot g(x) = 0\cdot\infty$ or $\infty\cdot 0$ we can use fractions $\left(f(x) = \frac{1}{1/f(x)}\right)$ to make the problem: $\lim_{x\to 0} \frac{g(x)}{1/f(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$

(C) $\infty - \infty$ - type

(vii) $\lim_{x\to 0^+} (\csc x - \cot x)$. $= \lim_{x\to 0^+}\left(\frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)}\right) = \lim_{x\to 0^+}\left(\frac{1-\cos(x)}{\sin(x)}\right) = \lim_{x\to 0^+}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{0}{1} = 0$

$\frac{1}{0} - \frac{1}{0}$ $\infty - \infty$ $\frac{1-1}{0}$ $\frac{0}{0}$

$\csc(x) = \frac{1}{\sin(x)}$

$\cot(x) = \frac{\cos(x)}{\sin(x)}$

2. Find the equations of all horizontal asymptotes of $y = \dfrac{3e^{3x} + 4e^{x} + 5}{2e^{3x} + e^{x} + 3}$.

horizontal asymptotes: $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$

$$\lim_{x\to\infty} \frac{3e^{3x}+4e^{x}+5}{2e^{3x}+e^{x}+3} \cdot \frac{e^{-3x}}{e^{-3x}}$$

$$= \lim_{x\to\infty} \frac{3e^{3x-3x}+4e^{x-3x}+5e^{-3x}}{2e^{3x-3x}+e^{x-3x}+3e^{-3x}}$$

$$= \lim_{x\to\infty} \frac{3+\cancel{4e^{-2x}}^{\,0}+\cancel{5e^{-3x}}^{\,0}}{2+\cancel{e^{-2x}}^{\,0}+\cancel{3e^{-3x}}^{\,0}}$$

$$= \frac{3}{2}$$

note. $\lim_{x\to\infty} \frac{1}{x} = 0$

think: 1/10, 1/100, 1/1000

Cheat sheet for $\frac{Ax^n + \ldots + a_1 x + a}{Bx^m + \ldots + b_1 x + b}$

Degree comparison:

numerator < denominator => $y=0$

numerator = denominator => $y = \frac{A}{B}$

numerator > denominator => none

3. Evaluate the following limits using L'Hopital's Rule where necessary.

(A) <u>0/0 - type, ∞/∞ - type and $0 \cdot \infty$ - type</u>

(i) $\lim_{x\to\infty} \dfrac{\ln(1+x)}{x}$.

(ii) $\lim_{x\to\infty} \dfrac{\sin(x)+\sin(2x)}{x^2+1}$.

(iii) $\lim_{x\to 0^+} x\ln(x)$.

(B) <u>1^∞ - type, ∞^0 - type and 0^0 - type</u>

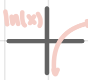
(iv) $\lim_{x\to\infty} (1+x)^{1/x}$.

(v) $\lim_{x\to\infty} \left(1 - \dfrac{2}{x}\right)^{x}$.

(vi) $\lim_{x\to 0^+} x^{x}$.

(i) $\lim_{x\to\infty} \frac{\ln(1+x)}{x}$

$$= \frac{\lim_{x\to\infty} \ln(1+x)}{\lim_{x\to\infty} x}$$



$= \frac{\infty}{\infty}$ L'H

$$= \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1}$$

$$= \lim_{x\to\infty} \frac{1}{1+x} = 0$$

think: 1/10, 1/100, 1/1000

(ii) $\lim_{x\to\infty} \frac{\sin(x)+\sin(2x)}{x^2+1}$

$$= \frac{\lim_{x\to\infty} \sin(x)+\sin(2x)}{\lim_{x\to\infty} x^2+1}$$

$$= \frac{\text{DNE} + \text{DNE}}{x^2+1}$$

Squeeze theorem:

$-1 \le \sin(x) \le 1$

$-1-1 \le \sin(x)+\sin(2x) \le 1+1$

$\frac{-2}{x^2+1} \le \frac{\sin(x)+\sin(2x)}{x^2+1} \le \frac{2}{x^2+1}$

$x^2+1>0$ always

(iii) $\lim_{x\to 0^+} x \cdot \ln(x)$

end behavior

$= 0 \cdot (-\infty)$ rewrite

$$= \lim_{x\to 0^+} \left(\frac{1}{1/x} \cdot \ln(x)\right)$$

$$= \lim_{x\to 0^+} \left(\frac{\ln(x)}{1/x}\right)$$

$\frac{1}{x} = x^{-1}$

$= \frac{\infty}{\infty}$ L'H

$$= \lim_{x\to 0^+} \left(\frac{1/x}{-1/x^2}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{1}{x} \cdot -x^2\right)$$

$$= \lim_{x\to 0^+} (-x) = 0$$

(iv) $\lim_{x\to\infty} (1+x)^{1/x}$

$\lim_{x\to\infty} (1+x) = \infty$

$\lim_{x\to\infty} \frac{1}{x} = 0$

$\Big\rangle\ \infty^0$

$$= \lim_{x\to\infty} e^{\ln((1+x)^{1/x})}$$

$$= \lim_{x\to\infty} e^{\frac{1}{x}\ln(1+x)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln(1+x)}{x}}$$

$= e^{\frac{\infty}{\infty}}$ L'H

$$= e^{\lim_{x\to\infty} \frac{\frac{1}{1+x}}{1}}$$

$$= e^{\lim_{x\to\infty} \frac{1}{1+x}} = e^0 = 1$$

(v) $\lim_{x\to\infty} \left(1-\frac{2}{x}\right)^x$

$= (1-0)^\infty = 1^\infty$

$$= \lim_{x\to 0} e^{\ln(1-\frac{2}{x})^x}$$

$= \lim_{x\to\infty} e^{x \cdot \ln(1-\frac{2}{x})}$ $\infty \cdot 0$

$$= e^{\lim_{x\to\infty} \frac{1}{1/x} \cdot \ln(1-\frac{2}{x})}$$

$$= e^{\lim_{x\to\infty} \frac{\ln(1-\frac{2}{x})}{1/x}}$$

$$= e^{\frac{\ln(1-1)}{0}} = e^{\frac{0}{0}}$$

$$= e^{\lim_{x\to\infty} \left(\frac{-2}{1-2/x}\right)} = e^{\left(\frac{-2}{1+0}\right)} = e^{-2}$$

(vi) $\lim_{x\to 0} x^x$

$= 0^0$

$$= \lim_{x\to 0} e^{\ln(x^x)}$$

$= \lim_{x\to 0} e^{x\ln(x)}$ $0 \cdot \infty$

$$= \lim_{x\to 0} e^{\frac{1}{1/x} \cdot \ln(x)}$$

$$= e^{\lim_{x\to 0} \frac{\ln(x)}{1/x}}$$

$$= e^{\frac{\infty}{\infty}}$$

$$= e^{\lim_{x\to 0} \left(\frac{1/x}{-1/x^2}\right)} = e^{\lim_{x\to 0}(-x)} = 1$$