Math 10350 – Example Set 11B

1. Sketch the graph of $g(x) = xe^{-x^2}$ by completing the steps below.

a. Find all x-intercepts and y-intercept of the graph of $g(x)$ whenever possible.

set $g(x)=0$

$0 = xe^{-x^2}$

$x=0 \quad e^{-x^2}=0$

never

set $x=0$

$y = 0e^{-(0)^2}$

$y=0$

intercepts:

$(0,0)$

b. Find coordinates of all critical points, vertical asymptotes, and places where $g(x)$ are undefined. $\left(g'(x) = (1-2x^2)e^{-x^2}\right)$

$g(x) = xe^{-x^2}$

$g'(x) = (1)e^{-x^2} + (e^{-x^2} \cdot (-2x)) \cdot x$

$f'(x)=0$ or DNE

$0 = e^{-x^2} - 2x^2 e^{-x^2}$

$0 = e^{-x^2}(1-2x^2)$

$0 = e^{-x^2} \quad 0 = 1-2x^2$

never $\quad 2x^2 = 1$

$x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}$

denominator $=0$

$g(x) = \frac{x}{e^{x^2}}$

$e^{x^2} = 0$

never

domain issues

none

c. Determine where $g(x)$ is increasing and where it is decreasing.

$f'(x) > 0$

$f'(x) < 0$

increasing: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

decreasing: $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

$f'(x) = e^{-x^2} - 2x^2 e^{-x^2}$

$f'(-1) = e^{-(-1)^2} - 2(-1)^2 e^{-(-1)^2}$

$= e^{-1} - 2e^{-1}$

$= -e^{-1}$

$f'(0) = e^{-(0)^2} - 2(0)^2 e^{-(0)^2}$

$= e^0 - 0$

$= 1$

$f'(1) = e^{-(1)^2} - 2(1)^2 e^{-(1)^2}$

$= e^{-1} - 2e^{-1}$

$= -e^{-1}$

find extrema (max & min):

$f\left(\frac{-1}{\sqrt{2}}\right) = \left(\frac{-1}{\sqrt{2}}\right) e^{-\left(\frac{-1}{\sqrt{2}}\right)^2}$

$= \left(\frac{-1}{\sqrt{2}}\right) e^{-\frac{1}{2}}$

$= \frac{-1}{\sqrt{2e}}$ minimum

$f\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right) e^{-\left(\frac{1}{\sqrt{2}}\right)^2}$

$= \left(\frac{1}{\sqrt{2}}\right) e^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{2e}}$ maximum

$f''(x) = 0$ or DNE

d. Determine the concavity and coordinates of inflection points of $g(x)$. $\left(g''(x) = (4x^3 - 6x)e^{-x^2}\right)$

$f'(x) = \underbrace{e^{-x^2}}_{f} - \underbrace{2x^2}_{g}\,\underbrace{e^{-x^2}}_{g}$

$f''(x) = -2xe^{-x^2} + [-4xe^{-x^2} + (-2xe^{-x^2})(-2x^2)]$

$0 = -2xe^{-x^2} - 4xe^{-x^2} + 4x^3e^{-x^2}$

$0 = -6xe^{-x^2} + 4x^3e^{-x^2}$

$0 = e^{-x^2}(-6x + 4x^3)$

$-\sqrt{\tfrac{3}{2}}$ (−, concave down), 0, (+, concave up), $\sqrt{\tfrac{3}{2}}$: sign chart $- \;|\; + \;|\; - \;|\; +$

$0 = e^{-x^2} \cdot x \cdot (4x^2 - 6)$

$0 = e^{-x^2}$ never

$0 = x$

$0 = 4x^2 - 6$

$6 = 4x^2$

$\frac{3}{2} = x^2$

$\pm\sqrt{\frac{3}{2}} = x$

$f''(x) = e^{-x^2}(4x^3 - 6x)$

$f''(-2) = e^{-(-2)^2} \cdot (4(-2)^3 - 6(-2))$
$= e^{-4} \cdot (-32 + 12)$ negative

$f''(-1) = e^{-(-1)^2}(4(-1)^3 - 6(-1))$
$= e^{-1}(-4 + 6)$ positive

$f''(1) = e^{-(1)^2}(4(1)^3 - 6(1))$
$= e^{-1}(4 - 6)$ negative

$f''(2) = e^{-(2)^2}(4(2)^3 - 6(2))$
$= e^{-4}(32 - 12)$ positive

$f(-\sqrt{\tfrac{3}{2}}) = -\sqrt{\tfrac{3}{2}}\,e^{-\frac{3}{2}}$ $\quad f(0) = 0$ $\quad f(\sqrt{\tfrac{3}{2}}) = \sqrt{\tfrac{3}{2}}\,e^{-\frac{3}{2}}$

e. Find all asymptotes and limit at infinity whenever applicable.

vertical: denominator $= 0$

horizontal: $\lim_{x \to \pm\infty} g(x) = L \Rightarrow y = L$

"end behavior"

no vertical asymptotes

horizontal:

$\lim_{x \to \infty} xe^{-x^2} = \lim_{x \to \infty} \frac{x}{e^{x^2}} \quad \frac{\infty}{\infty}$ L'H

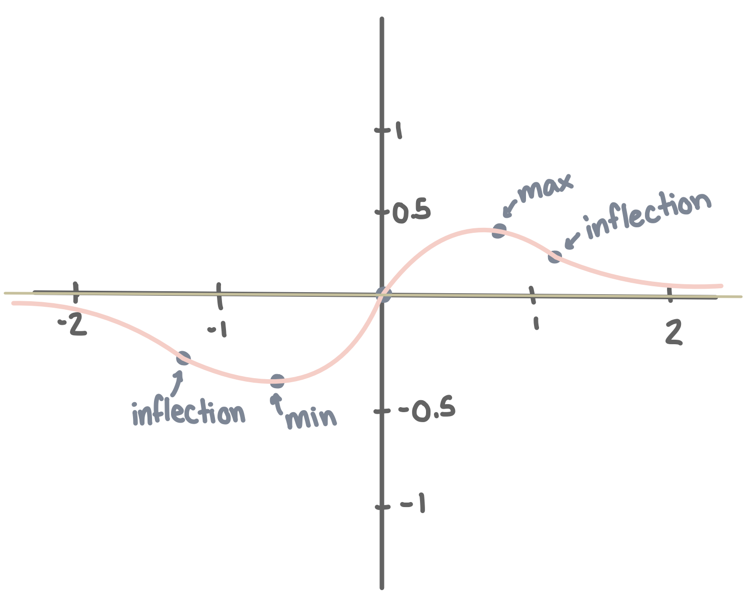
$= \lim_{x \to \infty} \frac{1}{2xe^{x^2}} = 0$

$\lim_{x \to -\infty} xe^{-x^2} = \lim_{x \to -\infty} \frac{x}{e^{x^2}} \quad \frac{\infty}{\infty}$ L'H

$= \lim_{x \to -\infty} \frac{1}{2xe^{x^2}} = 0$

graph approaches $y = 0$ on both ends

f. Sketch the graph below labeling all important features. Your picture should be large and clear.



Recall all points:

- critical points
  - ↳ max: $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}) \sim (0.7, 0.4)$
  - ↳ min: $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2e}}) \sim (-0.7, -0.4)$
- inflection points
  - ↳ down to up: $(-\sqrt{\tfrac{3}{2}}, -\sqrt{\tfrac{3}{2}}e^{-\frac{3}{2}}), (\sqrt{\tfrac{3}{2}}, \sqrt{\tfrac{3}{2}}e^{-\frac{3}{2}})$ $\sim (1.2, 0.3)$ $\sim (-1.2, -0.3)$
  - ↳ up to down: $(0, 0)$

Recall intervals:

- increasing: $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
- decreasing: $(-\infty, \frac{-1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)$
- concave up: $(-\sqrt{\tfrac{3}{2}}, 0) \cup (\sqrt{\tfrac{3}{2}}, \infty)$
- concave down: $(-\infty, -\sqrt{\tfrac{3}{2}}) \cup (0, \sqrt{\tfrac{3}{2}})$

Recall asymptotes:

- vertical: none
- end behavior: $y = 0$