Math 10350 – Example Set 12A

1a. Find the absolute (global) maximum and minimum of $f(x) = xe^{-x}$ on the interval $[0.5, 2]$. Write down the range of the values of $f(x)$ for $0.5 \le x \le 2$.

Critical Points

$f'(x) = (1)(e^{-x}) + (-e^{-x})(x)$

$f'(x) = e^{-x} - xe^{-x}$

$0 = e^{-x}(1-x)$

$0 = e^{-x}$ $\quad$ $0 = 1-x$

never $\quad$ $x = 1$

Test Points

$f(0.5) = \frac{1}{2}e^{-\frac{1}{2}} = \frac{1}{2\sqrt{e}} \approx 0.303$

$f(1) = 1e^{-1} = \frac{1}{e} \approx 0.368$

$f(2) = 2e^{-2} = \frac{2}{e^2} \approx 0.271$

Range

$\left[\frac{2}{e^2}, \frac{1}{e}\right]$

1b. Using the steps below, find the global maximum and minimum of $f(x) = xe^{-x}$ on $[0.5, \infty)$.

Step 1: Find all critical points in the domain of $f(x)$ and the values of $f(x)$ there. Classify them using first derivative test.

Critical Points

$f'(x) = (1)(e^{-x}) + (-e^{-x})(x)$

$f'(x) = e^{-x} - xe^{-x}$

$0 = e^{-x}(1-x)$

$0 = e^{-x}$ $\quad$ $0 = 1-x$

$x = 1$

f'(x) - sign line

$f'(x) = e^{-x}(1-x)$

always positive

pos. $x<1$
neg. $x>1$

+ pos · pos $\quad$ − pos · neg

0.5 $\quad$ 1

$f'(x)$ pos → neg at $x=1$

↳ $x=1$ is a maximum

Step 2: Find the values of $f(x)$ at the end-points (if any) of its domain. ___________________

Test Included Endpoints

$f(0.5) = \frac{1}{2}e^{-\frac{1}{2}} = \frac{1}{2\sqrt{e}} \approx 0.303$

Step 3: If end-point not included, or $\pm\infty$, find all limits of $f(x)$ towards end of interval.

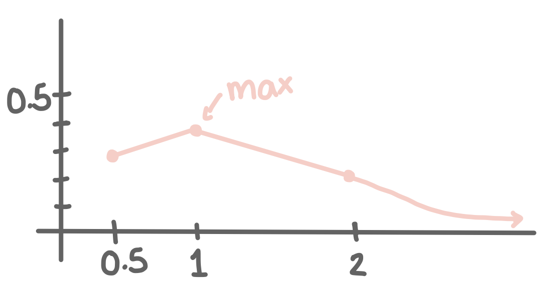
Test End Behaviors

$\lim\limits_{x\to\infty} xe^{-x} = \lim\limits_{x\to\infty} \frac{x}{e^x}$ $\quad \frac{\infty}{\infty}$ L'H

$= \lim\limits_{x\to\infty} \frac{1}{e^x} = 0$

we have to check open ends (parenthesis) because of asymptotes. What if the graph randomly goes to infinity? Think $\frac{1}{x}$

Step 4: Give a schematic sketch (ignore concavity) of the graph of $f(x)$ clearly indicating where the global maximum and minimum are. State the global maximum and minimum of $f(x)$ on $[0.5, \infty)$ if any. Find the range of $f(x)$ for x in $0.5 \le x < \infty$.

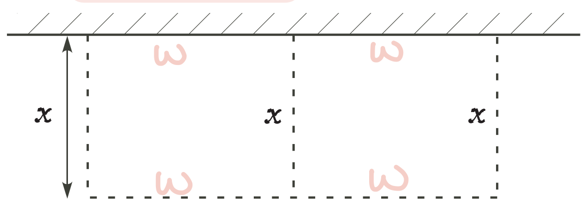


range: $(0, \frac{1}{e}]$

$f(1) = \frac{1}{e}$ so it is included

$\lim\limits_{x\to\infty} f(x) = 0$, but we can not say it is included

2. A landscaper plans to use 120 m of fencing and a very wide straight wall to make two rectangular enclosures with the same dimensions as shown.



fence $= 3x + 2w = 120 \Rightarrow 2w = 120 - 3x$

$w = \frac{120-3x}{2}$

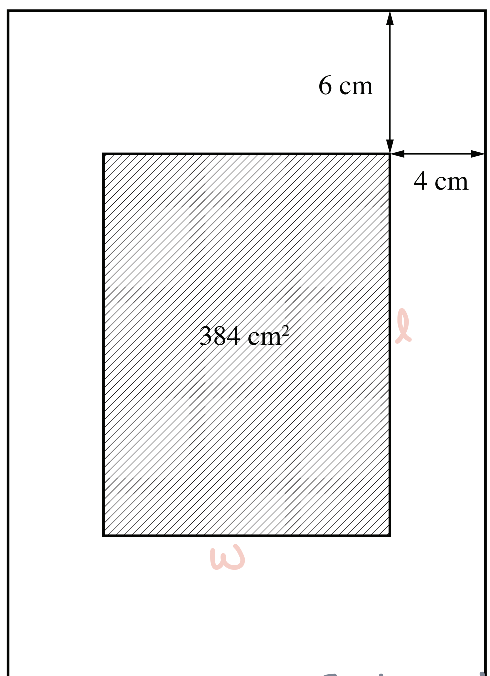
if $w = 0$, $3x = 120$
$x = 40$

a. Write down the possible values of x. $[0, 40]$

b. Find the maximum value of the total area of the enclosures. What are the dimensions of each enclosure when maximum occurs? Test end points: $x=0 \Rightarrow A=0$, $x=40 \Rightarrow w=0 \Rightarrow A=0$ Note that if $x=0$ or $w=0$ then $A=0$

$A = x(2w) \Rightarrow A = 2x\left(\frac{120-3x}{2}\right) = 120x - 3x^2 \Rightarrow \frac{dA}{dx} = 120 - 6x \Rightarrow 0 = 120 - 6x \Rightarrow x = 20$ $A(20) = 120(20) - 3(20)^2$
$= 1200$

3. The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area.



Given:

Printed $= 384 = l \cdot w$

$A_{poster} = (w+8)\underbrace{(l+12)}_{}$ 2·margin

Step 0: Use constraint to find possible values

$0 < w < \infty$ domain

not included as area = 384

w can be a fraction $w = \frac{1}{384} \Rightarrow l = (384)^2$

Step 1. maximize Area with constraint

Plug in constraint

$384 = l \cdot w$

$\frac{384}{w} = l$

$A = (w+8)\left(\frac{384}{w} + 12\right)$

$A = 384 + 12w + \frac{3072}{w} + 96$

$A = 12w + \frac{3072}{w} + 480$

Set $\frac{dA}{dw} = 0$

$\frac{dA}{dw} = 12 - \frac{3072}{w^2}$ $\longrightarrow$ DNE when $w = 0$

$0 = 12 - \frac{3072}{w^2}$

$\frac{3072}{w^2} = 12$

$\frac{3072}{12} = w^2$

$256 = w^2$

$\pm 16 = w$ ← $w = -16$ would imply negative distances

Plug in

$w = 16$ $\quad 384 = l \cdot (16)$

$24 = l$

Area $= (16+8)(24+12) = 60$

Step 2. Included End Points

none

Step 3. End Behavior

$\lim_{w \to 0} 12w + \frac{3072}{w} + 480 = \infty$

$\lim_{w \to \infty} \underbrace{12w + \frac{3072}{w} + 480}_{\text{area poster}} = \infty$

these are printed material dimensions and we want poster dimensions

Poster dimensions

$l + 12 = 24 + 12 = 36$

$w + 8 = 16 + 8 = 24$

36×24