Warning: If the domain is closed & bounded we must test endpoints!

1. A cylindrical can without a top is made to contain 100 cm^3 of liquid. Find the dimension that will minimize the cost of the material to make the can if the material for the side costs \$2/cm^2 and the material for the base costs \$3/cm^2.

2. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3cm and 4cm if two sides of the rectangle lie along the legs.

3. Show that of all the isosceles triangles with a perimeter of 30 cm, the one with the largest area is equilateral.

1.

Step 1: Minimize Cost

Plug in constraint

constraints are equal to a constant

Volume $= \pi r^2 h = 100 \implies h = \frac{100}{\pi r^2}$ ← Step 0: domain $0 < r < \infty$

r=0 implies V=0 contradiction

r=∞ implies h=0 implies V=0 contradiction

Surface Area = sides + bottom

= (circumference)·height + bottom area

$= (2\pi r)\cdot h + \pi r^2$

Cost = (side cost)(side surface) + (bottom cost)(bottom cost)

$= (2)(2\pi r h) + (3)(\pi r^2)$

$= 4\pi r\left(\frac{100}{\pi r^2}\right) + 3\pi r^2$

Set $\frac{dC}{dr} = 0$

$C = 400\frac{1}{r} + 3\pi r^2$

$\frac{dC}{dr} = -\frac{400}{r^2} + 6\pi r = 0$

$6\pi r = \frac{400}{r^2}$ ⟶ DNE $r=0$

would not give V = 100 out of domain

$6\pi r^3 = 400$

$r^3 = \frac{200}{3\pi}$

$r = \sqrt[3]{\frac{200}{3\pi}}$

200 = 2 · 100, 100 = 2 · 50, 50 = 2 · 25

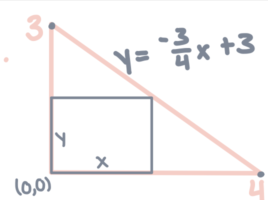
$r = 2\sqrt[3]{\frac{25}{3\pi}}$

Step 3: End Behaviors

$\lim_{r\to 0} 400\frac{1}{r} + 3\pi r^2 = \infty + 0 = \infty$

$\lim_{r\to \infty} 400\frac{1}{r} + 3\pi r^2 = 0 + \infty = \infty$

2.



maximize $A = x\cdot y$

constraint $y = -\frac{3}{4}x + 3$

domain: $0 \le x \le 4$

x=0 implies

x=4 implies y=0 implies

Area would then be zero which is okay but not maximal

Plug in constraint

$A = x\left(-\frac{3}{4}x + 3\right)$

$A = -\frac{3}{4}x^2 + 3x$

maximize $\frac{dA}{dx} = 0$

$\frac{dA}{dx} = -\frac{3}{2}x + 3 = 0$

$3 = \frac{3}{2}x$

$2 = x$

max area

$x = 2$ $\quad y = -\frac{3}{4}(2) + 3 = -\frac{3}{2} + 3 = \frac{3}{2}$

Max area:

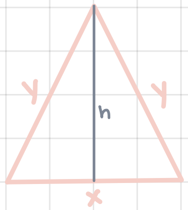
$2 \times \frac{3}{2} = 3\text{cm}^2$

Step 2: Included End Points

$A(0) = -\frac{3}{4}(0)^2 + 3(0) = 0$

$A(4) = -\frac{3}{4}(4)^2 + 3(4) = 0$

3. Show that of all the isosceles triangles with a perimeter of 30 cm, the one with the largest area is equilateral.



isosceles triangle: two sides are the same
equilateral triangle: same angles & side length

$h = \sqrt{y^2 - (\frac{x}{2})^2}$

straight line / bent in half
domain: $0 \le y \le 15$

box method:

$\times$	30	$-2y$
30	900	$-60y$
$-2y$	$-60y$	$+4y^2$

Given: $x + 2y = 30 \longrightarrow x = 30 - 2y$

Area $= \frac{1}{2}(\text{base})(\text{height})$

$= \frac{1}{2}x \cdot \sqrt{y^2 - (\frac{x}{2})^2}$

$A = \frac{1}{2}(30-2y) \cdot (y^2 - \frac{1}{4}(30-2y)^2)^{1/2}$ ← multiple chain rules

$= (15-y)(y^2 - \frac{1}{4}(900 - 120y + 4y^2))^{1/2}$

$= (15-y)(y^2 - 225 + 30y - y^2)^{1/2}$

$= (15-y)(30y-225)^{1/2}$

domain: $7.5 \le x \le 15$

Maximize Area

$\frac{dA}{dx} = (-1)(30y-225)^{1/2} + \frac{1}{2}(30y-225)^{-1/2}(30)(15-y)$

$0 = -(30y-225)^{1/2} + (15)(15-y)(30y-225)^{-1/2} \longrightarrow$ DNE $30y - 225 \le 0$ no negatives in square roots

$0 = -(30y-225) + 15(15-y)$ multiplied by $(30y-225)^{1/2}$

$30y \le 225$

$y \le 7.5$

$0 = -30y + 225 + 225 - 15y$

$0 = 450 - 45y$

$45y = 450$

$y = 10$

when $y = 7.5$

$x + 2(7.5) = 30$

$x + 15 = 30$

$x = 15$

when $y = 10$

$x + 2(10) = 30$

$x + 20 = 30$

$x = 10$

Maximums happen when $f'(x)$ goes from positive to negative. Normally we would test a number to the left and right, but $y \le 7.5$ is undefined.

Included End points

$A(7.5) = (15-7.5)\sqrt{30(7.5)-225} = (7.5)\sqrt{255-255} = 0$

$A(15) = (15-15)\sqrt{30(15)-225} = 0 \cdot \sqrt{450-225} = 0$

dimensions of isosceles triangle with maximum area is $y = 10, x = 0$
i.e. an equilateral triangle