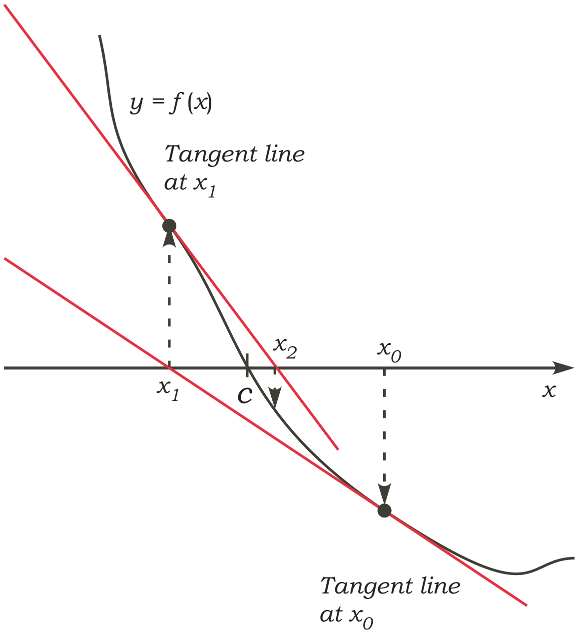
Math 10350 – Example Set 13A



Newton's Method
Steps to applying Newton's method to approximate the solution of $f(x) = 0$:

(1) Make an initial guess x_0 near to the zero you wish to find.

(2) Determine the new approximations x_1, x_2, ... :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(3) Check $|x_n - x_{n+1}| \to 0$ as $n \to \infty$ for convergence to required zero.

1a. Find $f'(x)$ if $f(x) = x^3 + x + 1$. Explain why we could see that $f(x)$ has a unique zero in the interval $[-1, 0]$.

1b. Apply Newton's Method with $x_0 = -0.5$ to estimate the zero of $f(x)$ up to three decimal places.

2. Estimate all solutions of $x^2 = \cos x$ up to four decimal places. (Hint: Sketch some graphs to see where the roots are located. Make your first guess for the root. You only need to find one.)

1(a). $f(x) = x^3 + x + 1$

$f'(x) = 3x^2 + 1$

$f(-1) = (-1)^3 + (-1) + 1$

$= -1 - 1 + 1 = -1$

$f(0) = (0)^3 + (0) + 1 = 1$

(b). $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Pick $x_0 = -\frac{1}{2}$.

$x_1 = -0.5 - \frac{f(-0.5)}{f'(-0.5)}$

$= -0.5 - \frac{(-\frac{1}{2})^3 + (-\frac{1}{2}) + 1}{3(-\frac{1}{2})^2 + 1}$

$= -0.5 - \frac{-\frac{1}{8} - \frac{1}{2} + 1}{3(\frac{1}{4}) + 1}$

$= -\frac{1}{2} - \frac{3/8}{7/4}$

$= -\frac{1}{2} - \frac{3}{8} \cdot \frac{4}{7}$

$= -\frac{1}{2} - \frac{3}{14}$

$= \frac{-7}{14} - \frac{3}{14} = \frac{-10}{14} = \frac{-5}{7}$

$-\frac{1}{8} - \frac{4}{8} + 1 = \frac{-5}{8} + 1 = \frac{3}{8}$

$\frac{3}{4} + \frac{4}{4} = \frac{7}{4}$

Intermediate Value Theorem.

Let $f(x)$ be a continuous function on the closed interval $[a,b]$. Let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a c in $[a,b]$ such that $f(c) = N$

↳ you can't get from $f(a)$ to $f(b)$ without going through $N = f(c)$.

Repeat

$x_2 = -\frac{5}{7} - \frac{(-\frac{5}{7})^3 + (-\frac{5}{7}) + 1}{3(-\frac{5}{7})^2 + 1} = -\frac{593}{868} \approx -0.6832$

$x_3 = -\frac{593}{868} - \frac{(-\frac{593}{868})^3 + (-\frac{593}{868}) + 1}{3(-\frac{593}{868}) + 1} \approx -0.6823$

$x_4 = (-0.6823) - \frac{(-0.6823)^3 + (-0.6823) + 1}{3(-0.6823) + 1} \approx -0.6823$

2. Estimate all solutions of $x^2 = \cos x$ up to four decimal places. (Hint: Sketch some graphs to see where the roots are located. Make your first guess for the root. You only need to find one.)

We have been asked to find solutions to $x^2 = \cos(x)$, but only have Newton's Method which finds a solution (root) to $f(x)=0$. We must first rearrange what has been given into what we need.

Given: $x^2 = \cos(x)$

$x^2 - \cos(x) = 0$

Needed: $f(x) = x^2 - \cos(x)$

Without a graph I would test a few points to try and get a positive and negative output:

$f(0) = (0)^2 - \cos(0) = -1$

$f(\pi/2) = (\pi/2)^2 - \cos(\pi/2)$

$= \frac{1}{4}\pi^2 - \frac{\sqrt{2}}{2} \approx 2.4674$

Pick the midpoint.

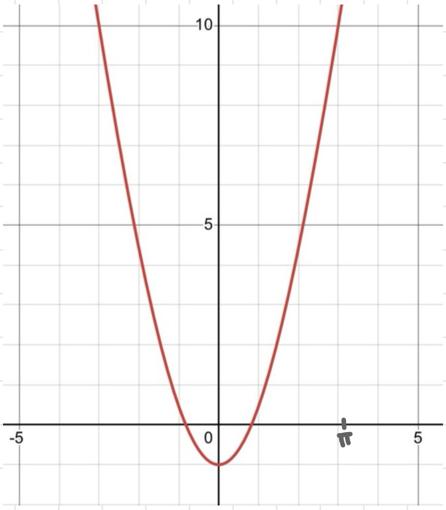
How to informed guess this graph:

parent function:

$f(x) = \underset{\text{stretch}}{a}x^2 + \underset{\text{shift}}{b}$

$-1 \leq \cos(x) \leq 1$

the cos(x) shifts the out put up or down less than 1 unit based on the input



First guess $x_0 = \pi/4$

$f'(x) = 2x + \sin(x)$

$x_1 = \frac{\pi}{4} + \frac{(\pi/4)^2 - \cos(\pi/4)}{2(\pi/4) + \sin(\pi/4)} \approx 0.8250$

$x_2 = 0.8250 + \frac{(0.8250)^2 - \cos(0.8250)}{2(0.8250) + \sin(0.8250)} \approx 0.8241$

$x_3 = 0.8241 + \frac{(0.8241)^2 - \cos(0.8241)}{2(0.8241) + \sin(0.8241)} \approx 0.8241$