▶ **Antiderivatives** (Reversing differentiation – Section 4.9)

Definition: We say that $F(x)$ is an **antiderivative** of $f(x)$ provided $F'(x) = f(x)$.

Example 1 Verify that $x^2 + 5$ is an antiderivative of $2x$. Can you write down a few more antiderivative of $2x$? What did you notice? Explain graphically.

$F(x) = x^2 + 5$

$F'(x) = 2x = f(x)$

We are looking for functions that derivative is $2x$. We know that the derivative of x^2 is $2x$. What can we add to x^2 that doesn't change the derivative? i.e. whats derivative is zero? $2x^2$? x^2+x? x^2+2?

Remark: We denote the family of antiderivatives of $2x$ by $x^2 + c$; $c \in \mathbb{R}$.

c is a real number

From Example 1, we see that

> **Theorem:** If $F(x)$ and $G(x)$ are antiderivatives of the same function throughout an interval, then they differ by a constant c over that interval; that is, for $a < x < b$
>
> $F'(x) = G'(x) \iff F(x) = G(x) + C$ for some number C.

Notation: If $F(x)$ is an antiderivative of $f(x)$, that is, $F'(x) = f(x)$. Then we may write

$$\int f(x)dx = F(x) + C$$

We call $\int f(x)dx$ the **indefinite integral**.

What can I take the derivative of to get k?

You can always check your answer by taking the derivative & making sure you get the original function

▶ **Basic indefinite integral formulas**

- For any constant k: $\int k\,dx \stackrel{?}{=} kx + C$. For Example: $\int 100\,dx \stackrel{?}{=} 100x + C$

  $F(x) = 100x + C \Rightarrow F'(x) = 100$

- Power Rule when $k \neq -1$: $\int x^k dx \stackrel{?}{=} \frac{1}{k+1} x^{k+1} + C$. For Example: $\int x^9 dx \stackrel{?}{=} \frac{1}{10} x^{10}$

  $F(x) = \frac{1}{10} x^{10} + C \Rightarrow F'(x) = x^9$

- Power Rule when $k = -1$: $\int \frac{1}{x} dx = \ln|x| + C$.

- Constant Multiple Rule: $\int kf(x)dx = k\int f(x)dx$, any k. For Example: $\int \frac{8}{x^2} dx \stackrel{?}{=} 8\int \frac{1}{x^2} dx$

  $= 8[-\frac{1}{x} + C]$

  $= -\frac{8}{x} + C_1$

  C is just a constant so we can make 8c into a new c

- Sum Rule: $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$.

  $F(x) = -8x^{-1} + C \Rightarrow F'(x) = 8x^{-2}$

- General Exponential function : $\int a^x dx \stackrel{?}{=} a^x \frac{1}{\ln(a)}$. For Example: $\int 10^x dx \stackrel{?}{=} \frac{1}{\ln(10)} 10^x$

  $F(x) = \frac{1}{\ln(10)} 10^x$

  $F'(x) = \frac{1}{\ln(10)} 10^x \cdot \ln(10) = 10^x$

- Exponential base e: $\int e^x dx \stackrel{?}{=} e^x$.

- Exponential function: $\int e^{ax} dx \stackrel{?}{=} \frac{1}{a} e^{ax}$. For Example: $\int e^{3x} dx \stackrel{?}{=} \frac{1}{3} e^{3x}$

  $F(x) = e^{3x} \Rightarrow F'(x) = e^{3x}$

1. Evaluate the following indefinite integrals:

a. $\int (1 + e^{2x} + e^2 + 3x - x^2)\,dx$

break it down:

$1 \rightarrow x$ constant (power rule)

$e^{2x} \rightarrow \frac{1}{2}e^{2x}$ exponential

$e^2 \rightarrow e^2 x$ constant

$3x \rightarrow \frac{3}{2}x^2$ power rule

$x^2 \rightarrow \frac{1}{3}x^3$ power rule

$= x + \frac{1}{2}e^{2x} + e^2 x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + C$

keep the sign

b. $\int \frac{2u^2 - 5u + \sqrt[3]{u}}{u^2}\,du$

$= \int 2 - \frac{5}{u} + \frac{u^{1/3}}{u^2}\,du$

$= \int 2 - 5u^{-1} + u^{-5/3}\,du$ $\quad -\frac{5}{3} + 1 = -\frac{5}{3} + \frac{3}{3} = -\frac{2}{3}$

$= 2u + 5\ln(u) - \frac{2}{3}u^{-2/3} + C \longleftarrow$ just include a $+C$ at the end of any anti-derivative

$F(u) = 2u + 5\ln(u) - \frac{3}{2}u^{-2/3} + C$

$F'(u) = 2 + 5 \cdot \frac{1}{u} - \frac{3}{2}\left(-\frac{2}{3}\right)u^{-5/3}$

$= 2 + \frac{5}{u} + u^{-5/3}$

2. Find the antiderivative F of function f satisfying the given condition:

$$f(x) = (e^x + 1)^2; \qquad F(0) = 3$$

In other words, solve the initial value problem:

$$\frac{dF}{dx} = (e^x + 1)^2; \qquad F(0) = 3$$

Step 1: antiderivative

$f(x) = (e^x + 1)^2 \longleftarrow$ we do not know how to do it from this point

$= e^{2x} + 2e^x + 1$

$F(x) = \frac{1}{2}e^{2x} + 2e^x + x + C$

$F'(x) = e^{2x} + 2e^x + 1$

Step 2: Solve for c

$F(0) = 3$ by given

$F(0) = \frac{1}{2}e^{2(0)} + 2e^{(0)} + (0) + C$

$3 = \frac{1}{2}e^0 + 2e^0 + C$

$3 = \frac{1}{2} + 2 + C$

$3 = \frac{5}{2} + C$

$\frac{1}{2} = C$

Step 3: Plug in c

$F(x) = \frac{1}{2}e^{2x} + 2e^x + x + C$

$= \frac{1}{2}e^{2x} + 2e^x + x + \frac{1}{2}$

3. A ball is projected upward from the ground with an initial velocity of 3 m/sec. Using calculus, write the velocity and position for the ball at time t. You may assume that the acceleration due to gravity is 10 m/s^2.

Given: $\underbrace{a(t) = 10 \text{ m/s}}_{\text{gravity}}$, $\underbrace{v(0) = 3 \text{ m/s}}_{\text{initial velocity}}$, $\underbrace{s(t) = 0}_{\text{from ground}}$

↳ $t=0$ ↳ $t=0$ ↳ $s(t)=0$

This is secretly <u>two</u> initial value problems

Recall: $s(t)$ = position

$v(t) = s'(t)$ = velocity

$a(t) = s''(t)$ = acceleration

Initial value Problem 1:

Step 1: antiderivative

$a(t) = 10$

$v(t) = 10t + C$

Step 2: Solve for c

$3 = v(0) = 10(0) + C$

$3 = C$

Step 3: Plug in c

$v(t) = 10t + 3$

Initial value Problem 2:

Step 1: Antiderivative

$v(t) = 10t + 3$

$s(t) = 5t^2 + 3t + C$

Step 2: Solve for c

$0 = s(0) = 5(0)^2 + 3(0) + C$

$0 = C$

Step 3: Plug in c

$s(t) = 5t^2 + 3t + 0$

Check:

$s(t) = 5t^2 + 3t$

$v(t) = s'(t) = 10t + 3$

$a(t) = s''(t) = 10$