Math 10350 Example Set 14A

1. Write down all integration (anti-differentiation) formula for the trigonometric functions.

Since derivatives and antiderivatives are inverses, we can always use our derivative definitions to build antiderivatives.

Trig. Derivatives

$\frac{d}{dx}[\sin(x)] = \cos(x)$ $\quad \frac{d}{dx}[\cos(x)] = -\sin(x)$ $\quad \frac{d}{dx}[\tan(x)] = \sec^2(x)$ $\quad \frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

Trig. Antiderivatives

$\int \cos(x)dx = \sin(x)+c$ $\quad \int \sin(x)dx = -\cos(x)+c$ $\quad \int \sec^2(x)dx = \tan(x)+c$ $\quad \int \sec(x)\tan(x)dx = \sec(x)+c$

take the antiderivative — w.r.t. x — include all translations — sign correction

2. A small piece of wood is bobbing up and down on the surface of a pond with its acceleration given by

$$a(t) = (\sin t - \cos t) \quad \text{m/s}^2.$$

Given that the cork has velocity 1 m/s and position -2 m when $t = \pi$ seconds, answer the following questions:

a. If $s(t)$ is the position of the cork, write in terms of s and its derivatives, a differential equation, and initial value conditions modeling the position of the cork.

b. Solve the equation in (a) for $s(t)$ by first finding for $s'(t)$.

3. Evaluate the following indefinite integrals:

a. $\int \frac{\tan\theta}{\cos\theta}\,d\theta = \int \tan(\theta)\sec(\theta)d\theta$

$= \sec(\theta)+c$

$\frac{d}{dx}[\sec(\theta)+c] = \tan(\theta)\sec(\theta)$

$= \tan(\theta)\cdot\frac{1}{\cos(\theta)}$

$= \frac{\tan(\theta)}{\cos(\theta)}$

b. $\int \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta = \int \frac{1}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)}\,dx$

$= \int \sec^2(x) + 1\,dx$

$= \tan(x) + x + c$

$\frac{d}{dx}[\tan(x)+x+c] = \sec^2(x)+1$

$= \frac{1}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)}$

$= \frac{1+\cos^2(x)}{\cos^2(x)}$

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b. Solve the equation in (a) for $s(t)$ by first finding for $s'(t)$.

(a) $s''(t) = \sin(t) - \cos(t)$; $s'(\pi) = 1$, $s(t) = \pi$

$s(t) = \int [\int \sin(t) - \cos(t)\, dt]\, dt$; $s'(\pi) = 1$, $s(\pi) = -2$

(b) $s'(t) = \int s''(t)\, dt$

$= \int \sin(t) - \cos(t)\, dt$

$= -\cos(t) - \sin(t) + c$

$1 = s'(\pi) = -\cos(\pi) - \sin(\pi) + c$

$1 = -(-1) - (0) + c$

$1 = 1 + c$

$0 = c$

$s'(t) = -\cos(t) - \sin(t) + 0$

$s(t) = \int s'(t)\, dt$

$= \int -\cos(t) - \sin(t)\, dt$

$= -\sin(t) + \cos(t) + c$

$-2 = s(\pi) = -\sin(\pi) + \cos(\pi) + c$

$-2 = -(0) + (-1) + c$

$-2 = -1 + c$

$-1 = c$

$s(t) = -\sin(t) + \cos(t) - 1$

Check:

$s(t) = -\sin(t) + \cos(t) - 1$

$s'(t) = -\cos(t) - \sin(t) + 0$

$s''(t) = \sin(t) - \cos(t)$ ✓

$s(\pi) = -(0) + (-1) - 1 = -2$ ✓

$s'(\pi) = -(-1) - 0 = 1$ ✓

Summation Notation (Section 5.1)

1. Assuming that the pattern in the sums below, write down (a) the formula for the general term, and (b) the sum using summation notation.

a. $\frac{5}{1+1^2} + \frac{5}{1+2^2} + \frac{5}{1+3^2} + \ldots + \frac{5}{1+15^2}$

n^{th} term

$a_n = \frac{5}{1+n^2}$

start at 1

$\sum_{n=1}^{15} \frac{5}{1+n^2}$

alternatively: Start at 0

$\sum_{n=0}^{14} \frac{5}{1+(n+1)^2}$

b. $\frac{5}{1+5^2} + \frac{5}{1+6^2} + \frac{5}{1+7^2} + \ldots + \frac{5}{1+13^2}$

$a_n = \frac{5}{1+(n+4)^2}$

I want $a_1 = \frac{5}{1+5^2}$

$\sum_{n=1}^{9} \frac{5}{1+(n+4)^2}$

$\sum_{n=0}^{8} \frac{5}{1+(n+5)^2}$

c. $\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{0}{n}\right)^2} + \left(\frac{1}{n}\right)\sqrt{1-\left(\frac{1}{n}\right)^2} + \left(\frac{1}{n}\right)\sqrt{1-\left(\frac{2}{n}\right)^2} + \ldots + \left(\frac{1}{n}\right)\sqrt{1-\left(\frac{n-1}{n}\right)^2}$

$a_i = \left(\frac{1}{n}\right)\sqrt{1-\left(\frac{i}{n}\right)^2}$ $\quad \sum_{i=1}^{n}\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{i-1}{n}\right)^2}$ $\quad \sum_{i=0}^{n-1}\left(\frac{1}{n}\right)\sqrt{1-\left(\frac{i}{n}\right)^2}$

Properties of summation notation:

(A) $\sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ $\qquad$ (B) $\sum_{k=1}^{n}(c \cdot a_k) = c \cdot \left(\sum_{k=1}^{n} a_k\right)$ $\qquad$ (C) $\sum_{k=1}^{n}(c) = c \cdot n$

Why are these properties true?

2. If $a_0 = 2$, $a_1 = 0$, $a_2 = -1$, $a_3 = -2$, and $a_4 = 0$. Find the value of the sums:

(a) $\sum_{j=2}^{4}(2a_j+3)$ $\qquad$ (b) $\sum_{n=0}^{2}\cos(a_n\pi)$.

(A) $\sum_{k=1}^{n}(a_k+b_k) = (a_1+b_1)+(a_2+b_2)+\ldots+(a_n+b_n) = (a_1+a_2+\ldots+a_n)+(b_1+b_2+\ldots+b_n) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(B) $\sum_{k=1}^{n}(c \cdot a_k) = c \cdot a_1 + c \cdot a_2 + \ldots + c \cdot a_n = c \cdot (a_1+a_2+\ldots+a_n) = c \cdot \sum_{k=1}^{n} a_k$

(C) $\sum_{k=1}^{n}(c) = \underbrace{c+c+\ldots+c}_{n \text{ times}} = c(\underbrace{1+1+\ldots+1}_{n \text{ times}}) = c \cdot n$

2.(a) $\sum_{j=2}^{4}(2a_j+3) = (2a_2+3)+(2a_3+3)+(2a_4+3)$

$= 2(a_2+a_3+a_4)+9$

$= 2(-1-2+0)+9$

$= 2(-3)+9$

$= -6+9$

$= 3$

(b) $\sum_{n=0}^{2}\cos(a_n \cdot \pi) = \cos(a_0 \cdot \pi)+\cos(a_1 \cdot \pi)+\cos(a_2 \cdot \pi)$

$= \cos(2\pi)+\cos(0 \cdot \pi)+\cos(-\pi)$

$= 1+1+(-1)$

$= 1$