Math 10350 – Example Set 15A
(Section 5.1 & 5.2)

(5.1) Right-endpoint Approximation. Estimate the area under the graph of $f(x) = e^{-x^2}$ over the interval $0 \leq x \leq 2$ using **4th right-endpoint approximation** (ie. with four sub-intervals). (Text notation: R_4).

How big should my step (Δx) be?

$\Delta x = \frac{b-a}{n}$ where $a \leq x \leq b$ and n = number of sub-intervals

$= \frac{2-0}{4} = \frac{1}{2}$

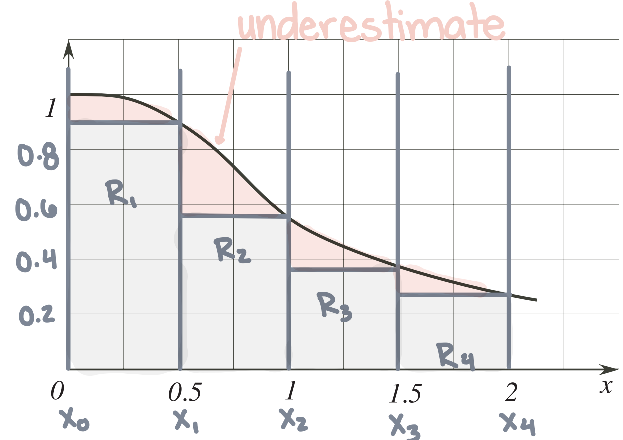
Right-endpoint means using right-hand value as the height of the rectangle:

$R_1 = f(x_1) \cdot \Delta x = (0.9)(\frac{1}{2})$

$R_2 = f(x_2) \cdot \Delta x = (0.55)(\frac{1}{2})$

$R_3 = f(x_3) \cdot \Delta x = (0.35)(\frac{1}{2})$

$R_4 = f(x_4) \cdot \Delta x = (0.25)(\frac{1}{2})$



General formula: $\sum_{i=0}^{n-1} f(x_{i+1}) \cdot \Delta x$
(right-endpoint)
← n counts
right-hand

(5.1) Left-endpoint Approximation. Estimate the area under the graph of $f(x) = e^{-x^2}$ over the interval $0 \leq x \leq 2$ using **4th left-endpoint approximation** (ie. with four sub-intervals). (Text notation: L_4).

How big should my step (Δx) be?

$\Delta x = \frac{b-a}{n}$ where $a \leq x \leq b$ and n = number of sub-intervals

$= \frac{2-0}{4} = \frac{1}{2}$

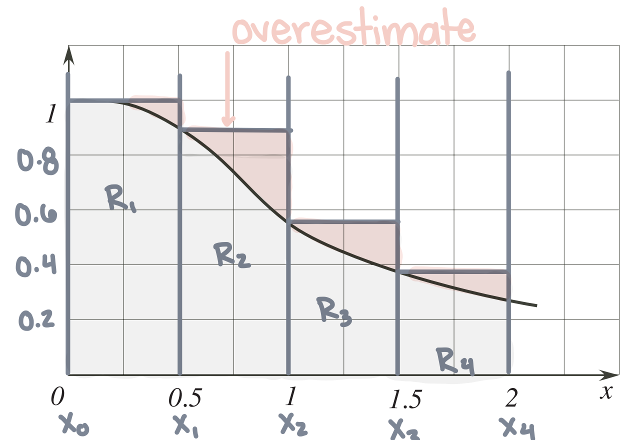
Left-endpoint means using left-hand value as the height of the rectangle:

$R_1 = f(x_0) \cdot \Delta x = (1)(\frac{1}{2})$

$R_2 = f(x_1) \cdot \Delta x = (0.9)(\frac{1}{2})$

$R_3 = f(x_2) \cdot \Delta x = (0.55)(\frac{1}{2})$

$R_4 = f(x_3) \cdot \Delta x = (0.39)(\frac{1}{2})$



General formula: $\sum_{i=0}^{n-1} f(x_i) \cdot \Delta x$
(left-endpoint)
← n counts
left-hand

(5.1) Midpoint Approximation. Estimate the area under the graph of $f(x) = e^{-x^2}$ over the interval $0 \leq x \leq 2$ using **Midpoint Rule** with four sub-intervals. (Text notation: M_4).

How big should my step (Δx) be?

$\Delta x = \frac{b-a}{n}$ where $a \leq x \leq b$ and n = number of sub-intervals

$= \frac{2-0}{4} = \frac{1}{2}$

Midpoint means using the midpoint value as the height of the rectangle:

$R_1 = f(x_{\frac{1}{2}}) \cdot \Delta x = (1)(\frac{1}{2})$

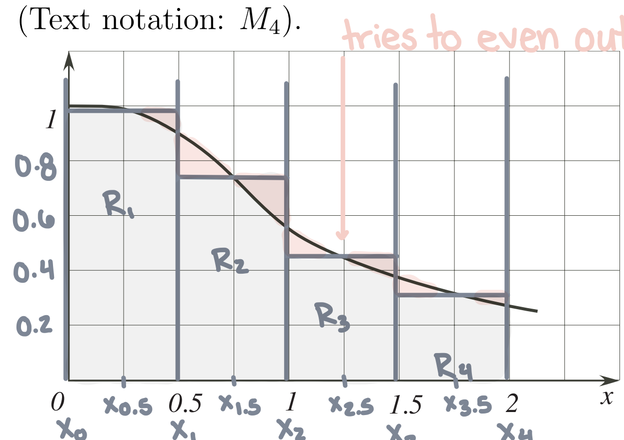
$R_2 = f(x_{1.5}) \cdot \Delta x = (0.65)(\frac{1}{2})$

$R_3 = f(x_{2.5}) \cdot \Delta x = (0.45)(\frac{1}{2})$

$R_4 = f(x_{3.5}) \cdot \Delta x = (0.3)(\frac{1}{2})$

$x_{n+\frac{1}{2}} = \frac{x_{n+1} - x_n}{2}$

so $x_{\frac{1}{2}} = \frac{x_1 - x_0}{2}$



General formula: $\sum_{i=0}^{n-1} f(x_{i+\frac{1}{2}}) \cdot \Delta x$
↑ slight joke

1. Using the **Nth right-endpoint approximation**, express the area under the graph of $f(x) = e^{-x^2}$ over $0 \le x \le 2$ as a limit of right-endpoint approximations.

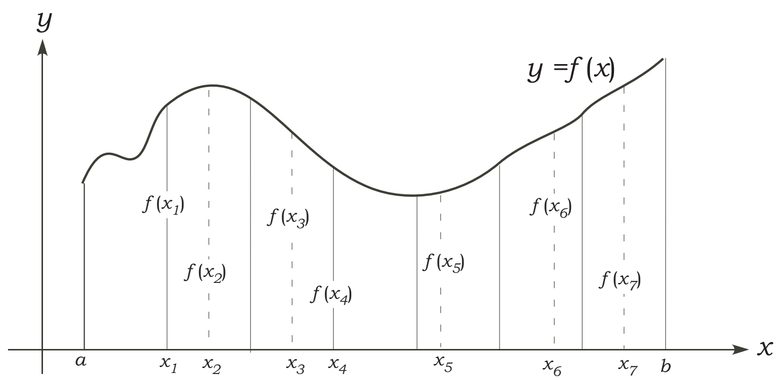
General formula: $\sum_{i=0}^{n-1} f(\underbrace{x_{i+1}}_{\text{right-hand}}) \cdot \Delta x$
(right-endpoint)

Ideally $n \to \infty$ that way we do not over/under-estimate:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_{i+1}) \cdot \Delta x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_{i+1}) \cdot \underbrace{\frac{b-a}{n}}_{\text{goes to 0}} = \lim_{n\to\infty} \sum_{i=0}^{n-1} e^{-(x_{i+1})^2} \cdot \left(\frac{b-a}{n}\right)$$

Remark. We denote the area under the graph of $f(x) = e^{-x^2}$ over $0 \le x \le 2$ with the definite integral notation: $\int_a^b f(x)\,dx = \int_0^2 e^{-x^2}dx$

Definite Integral of Positive Value functions. In general, we many select any point in a subinterval and do the same construction to obtain the area under the graph of $f(x)$.



These more general sums are called Riemann Sum. They give us a similar limiting formula for the value of the definite integral for a positive valued $f(x)$ over $[a, b]$. Write down the relation below:

Area $= \sum_{x=0}^{n-1} f(x)\,\Delta x$

Careful distinction: area under the graph refers to a rectangle w/ height f(x) or really (f(x)−0) is the height. This leads to negative areas under the graph! Areas between a graph & the x-axis are positive.

Riemann Sum for Continuous Functions

We have been computing Riemann sum for positive valued function up till now. These sums for positive functions estimates the area under their graphs over an interval.

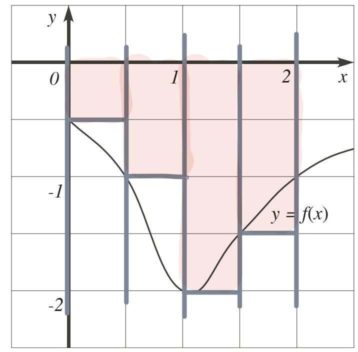
We could carry out the same computations for continuous functions in general.

2a. Find the Riemann sum for $f(x)$ over $[0, 2]$ using **4 equal subintervals** and the left endpoints.

Area $= \sum_{i=0}^{3} f(x_i)\cdot\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{i=0}^{3} f(x_i) = \frac{1}{2}\left[f(0)+f(0.5)+f(1)+f(1.5)\right]$

$= \frac{1}{2}\left[(-\frac{1}{2})+(-1)+(-2)+(-1.5)\right] = \frac{1}{2}\left[-\frac{1}{2}-1-2-1.5\right]$

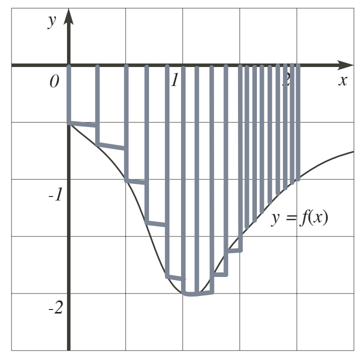
$= \frac{1}{2}[-5] = -\frac{5}{2}$



What value would you obtain if you allow more and more subinterval?

$\int_0^2 f(x)\,dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\cdot\Delta x$

$\frac{1}{n}$ i.e. goes to zero

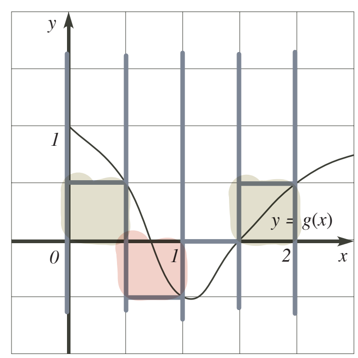


2b. Find the Riemann sum for $g(x)$ over $[0, 2]$ using **4 equal subintervals** and the right endpoints.

The drawing starts to make less sense as we switch from positive to negative during a sub-interval & have f(x)=0 meaning a rectangle of height zero:

Area $= \sum_{i=0}^{3} f(x_{i+1})\cdot\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{i=0}^{3} f(x_{i+1}) = \frac{1}{2}\left[f(\tfrac{1}{2})+f(1)+f(1.5)+f(2)\right]$

$= \frac{1}{2}\left[\frac{1}{2}+(-\frac{1}{2})+0+\frac{1}{2}\right] = \frac{1}{2}\left[\frac{1}{2}-\frac{1}{2}+0+\frac{1}{2}\right] = \frac{1}{2}\left[\frac{1}{2}\right] = \frac{1}{4}$



What value would you obtain if you allow more and more subinterval?

$\int_0^2 f(x)dx = \lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_{i+1})\cdot\Delta x$

As $n\to\infty$, it is hard to differentiate between left, right, and midpoint so we just use $\int_a^b f(x)\,dx$ for all limits.