Math 10350 – Example Set 15B
(Section 5.2, 5.4, 5.5, 5.6 & 5.7)

left hemisphere

1. Use geometry to compute the definite integral $\int_{-5}^{0} \sqrt{25 - x^2}\, dx$ $\longrightarrow$ $y = \sqrt{25 - x^2}$

$y^2 = 25 - x^2$

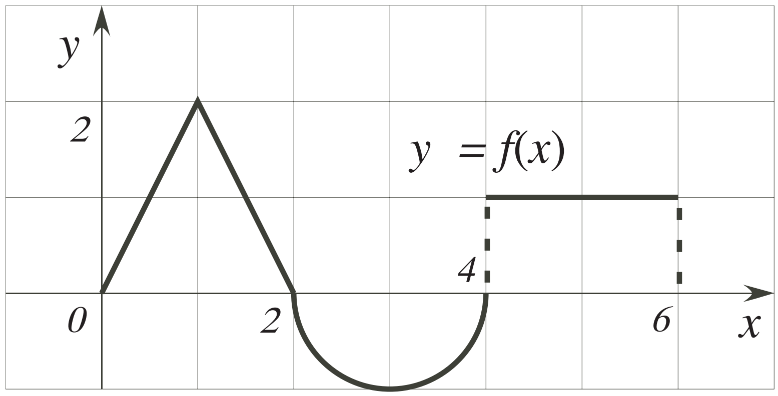
$x^2 + y^2 = 25$

i.e. circle of radius 5

positive semi-circle

$\int_{-5}^{0} \sqrt{25 - x^2}\, dx$ is the area bounded by the left-upper quarter

$= \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(5)^2 = \frac{25}{4}\pi$



2. Consider the graph of $f(x)$ above. Using geometry, find the value of all the definite integrals below:

the hemisphere has negative area under the graph

a. $\int_{0}^{2} f(x)\, dx \stackrel{?}{=} \triangle = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$

c. $\int_{2}^{4} f(x)\, dx \stackrel{?}{=} -⌓ = -\frac{1}{2}\pi r^2 = -\frac{1}{2}\pi(1)^2 = -\frac{1}{2}\pi$

b. $\int_{1}^{4} f(x)\, dx \stackrel{?}{=} \triangle - ⌓ = 2 - \frac{1}{2}\pi$

d. $\int_{0}^{6} f(x)\, dx \stackrel{?}{=} \triangle - ⌓ + \square = 2 - \frac{1}{2}\pi + (2)(1) = 4 + \frac{1}{2}\pi$

Properties of Definite Integral (5.2). Let $a < b < c$ and k be real numbers. Let $f(x)$ and $g(x)$ be continuous functions. Then we have the following:

i. $\int_{a}^{b} [f(x) + g(x)]\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

ii. $\int_{a}^{b} k \cdot f(x)\, dx = k \cdot \int_a^b f(x)dx$

iii. $\int_{a}^{b} f(x)\, dx + \int_{b}^{c} f(x)\, dx = \int_a^c f(x)dx$

We also define:

iv. $\int_{a}^{a} f(x)\, dx = 0$

v. $\int_{b}^{a} f(x)\, dx = -\int_a^b f(x)dx$

3. Given that $\int_{0}^{2} f(x)dx = \int_{2}^{3} f(x)dx = 5$, find

a. $\int_{0}^{3} f(x)dx \stackrel{?}{=} \int_0^2 f(x)dx + \int_2^3 f(x)dx = 5 + 5 = 10$

c. $\int_{0}^{2} f(x)dx + \int_{3}^{2} f(x)dx \stackrel{?}{=} \int_0^2 f(x)dx - \int_2^3 f(x)dx$

$= 5 - 5 = 0$

b. $\int_{0}^{2} [4f(x) + 2]dx \stackrel{?}{=} 4\int_0^2 f(x)dx + \int_0^2 2dx$

$= 4(5) + \int_0^2 2dx \leftarrow$ we don't know how to do this

Fundamental Theorem of Calculus (5.4). Let $F(x)$ be an anti-derivative of $f(x)$. Then

$$\int_a^b f(x)\,dx = \underline{F(b)-F(a)} = \underline{F(x)\big|_a^b} \quad \text{(shorthand notation).}$$

In other words: Note that F(x) includes a plus c, this disappears as F(b) would have plus c and - F(a) would have -(plus c)

Total change in $F(x)$ over $[a, b]$ $=$ $\underline{F(b)-F(a)}$ $=$ $\underline{\int_a^b f(x)dx}$

4. Evaluate the following definite integrals:

a. $\displaystyle\int_{-1}^{0} (1 + 3x - e^{-x})\,dx$

$F(x) = x + \frac{3}{2}x^2 + e^{-x} + C$

$F'(x) = 1 + 3x - e^{-x} = f(x)$

$\int_{-1}^{0} 1 + 3x - e^{-x}\,dx = F(0) - F(-1)$

$= x + \frac{3}{2}x^2 + e^{-x}\big|_{-1}^{0}$

$= 0 + \frac{3}{2}(0)^2 + e^{-0} - \left((-1) + \frac{3}{2}(-1)^2 + e^{-(-1)}\right)$

$= 0 + 0 + 1 + 1 - \frac{3}{2} + e$

$= \frac{1}{2} + e$

b. $\displaystyle\int_{\pi/2}^{\pi} \cos\theta\,d\theta$

$F(\theta) = \sin(\theta) + C$

$F'(\theta) = \cos(\theta)$

$\int_{\pi/2}^{\pi} \cos(\theta)\,d\theta = \sin(\theta)\big|_{\pi/2}^{\pi}$

$= \sin(\pi) - \sin(\pi/2)$

$= 0 - 1$

$= -1$

c. $\displaystyle\int \sqrt{x-1}\,dx = \int (x-1)^{1/2}dx$

The format suggest power rule:

$\int ax^n = \frac{a}{n+1}x^{n+1} + C$

But we have $(x-1)^{1/2}$ and not $x^{1/2}$

Try: $F(x) = \frac{1}{3/2}(x-1)^{3/2} + C = \frac{2}{3}(x-1)^{3/2} + C$

$F'(x) = \frac{2}{3}\cdot\frac{3}{2}(x-1)^{1/2} = (x-1)^{1/2}$

$\int \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2} + C$

d. $\displaystyle\int_{1}^{5} \sqrt{x-1}\,dx$

$F(x) = \frac{2}{3}(x-1)^{3/2} + C$

$F'(x) = (x-1)^{1/2}$

$\int_1^5 \sqrt{x-1}\,dx = \frac{2}{3}(x-1)^{3/2}\big|_1^5$

$= \frac{2}{3}(5-1)^{3/2} - \frac{2}{3}(0)^{3/2}$

$= \frac{2}{3}[(4)^{1/2}]^3 - 0$

$= \frac{2}{3}(2)^3$

$= \frac{16}{3}$