Math 10350 – Example Set 16A
Sections 5.5, 5.6 & 5.7

Second Fundamental Theorem of Calculus (5.5) If f is continuous on an open interval I containing a then, for all x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

1. Show that $\dfrac{d}{dx}\left[\displaystyle\int_a^{g(x)} f(t)\,dt\right] = f(g(x))g'(x)$

Recall FTC: $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$
Thus $\int_a^{g(x)} f(t)dt = F(g(x)) - F(a)$ where $F(a)$ is a constant. Therefore $\frac{d}{dx}\left[\int_a^{g(x)} f(t)dt\right] = \frac{d}{dx}\left[F(g(x)) - F(a)\right]$

$= F'(g(x))\cdot g'(x) - 0$

$= f(g(x))\cdot g'(x)$.

Hint: Let $H(x) = \displaystyle\int_a^x f(t)\,dt$. Then $H(g(x)) = \displaystyle\int_a^{g(x)} f(t)\,dt$. Compute $\dfrac{d}{dx}[H(g(x))]$.

2. Find the derivative of each of the following functions

a. $g(x) = \displaystyle\int_2^x t^2 \sin t\,dt$

$\frac{d}{dx}[g(x)] = x^2 \sin(x)$

b. $y = \displaystyle\int_1^{\cos x} (u + \sin u)\,du$

$g(x) = \cos(x)$
$f(x) = u + \sin(u)$

$\frac{d}{dx}[y] = [\cos(x) + \sin(\cos(x))]\cdot(-\sin(x))$

$= -\sin(x)\cos(x) - \sin(x)\sin(\cos(x))$

c. $F(x) = \displaystyle\int_x^{\sqrt{x}} \cos(t^2)\,dt$

$\int_{g(x)}^{h(x)} f(t)\,dt = F(h(x)) - F(g(x))$

$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)dt\right] = \frac{d}{dx}\left[F(h(x)) - F(g(x))\right]$

$= F'(h(x))\cdot h'(x) - F'(g(x))g'(x)$

$= f(h(x))\cdot h'(x) - f(g(x))g'(x)$

$\frac{d}{dx}\left[\int_x^{\sqrt{x}} \cos(t^2)\,dt\right]$

$= \cos((\sqrt{x})^2)\cdot(\frac{1}{2}x^{-1/2}) - \cos((x)^2)\cdot(1)$

$= \frac{1}{2\sqrt{x}}\cos(x) - \cos(x^2)$

Alternative: $\int_x^{\sqrt{x}} \cos(t^2)dt = \int_x^0 \cos(t^2)dt + \int_0^{\sqrt{x}} \cos(t^2)dt$

$= -\int_0^x \cos(t^2)dt + \int_0^{\sqrt{x}} \cos(t^2)dt$

$\frac{d}{dx}\left[-\int_0^x \cos(t^2)dt + \int_0^{\sqrt{x}} \cos(t^2)dt\right]$

$= -(\cos(x^2)\cdot 1) + \cos((\sqrt{x})^2)\ \frac{1}{2\sqrt{x}}$

$= -\cos(x^2) + \frac{1}{2\sqrt{x}}\cos(x)$

3. Water flows into a large tank at rate $r(t)$ liters/min given in the table below. If the initial volume of water is 100 liters , estimate the volume of water in the tank at $t = 4$ minutes using **left-endpoint** approximation.

$\int_a^b r(t)dt = V(b) - V(a)$

$\int_0^4 r(t)dt = \sum_{i=0}^{4} r(t)\Delta t$

t	0	1	2	3	4	5	6
$r(t)$	10	15	18	20	23	21	25

Note $\int_a^b r(t)dt$ is the change in the volume. We must add the initial back.

$= r(0)+r(1)+r(2)+r(3) = 10+15+18+20 = 63$

$V(4) - V(0) = 63$

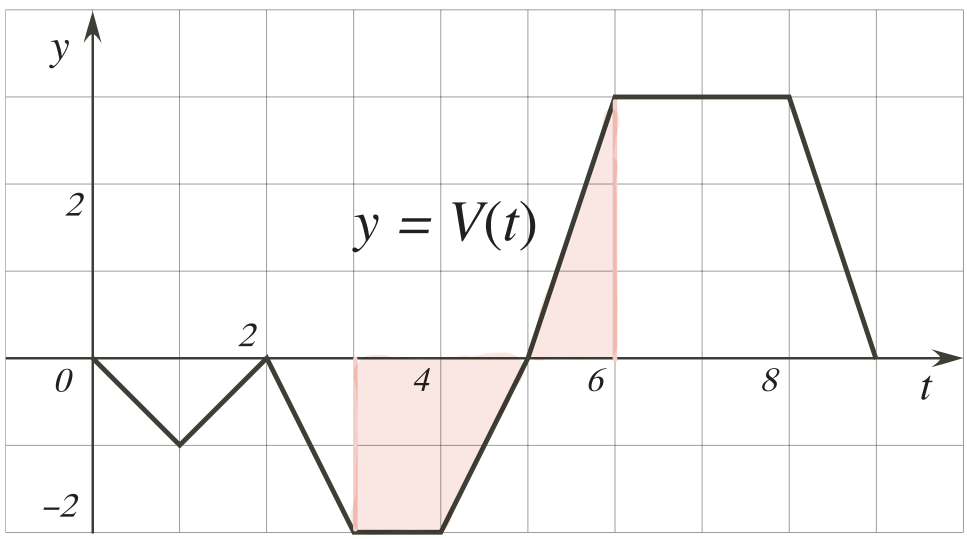
$V(4) = 63 + V(0) = 163$

4. The graph of the velocity V of a particle moving on a horizontal straight line is given below. Let $S(t)$ meters be the displacement (position) of the particle after time t minutes. Assume that $S(0) = 2$. Find the exact value of the following quantities.

$S(b) - S(a)$

a. The change in the displacement of the particle over the duration $[3, 6]$.

b. The displacement of the particle after 2 minutes.



(a) change in displacement $= S(b) - S(a) = \int_a^b s'(t)dt = \int_a^b v(t)dt$ = area under the curve

$S(6) - S(3) = \int_3^6 s'(t)dt = -\square - \triangle + \triangle$

$= (1)(-2) + \frac{1}{2}(1)(-2) + \frac{1}{2}(1)(3)$

$= -2 - 1 + \frac{3}{2}$

$= -3 + \frac{3}{2}$

$= \frac{3}{2}$

Note: a negative velocity is moving in the negative direction (left) so this particle is moving left from time 0 to 5 and right from time 5 to 9. Displacement only cares about start and end point so moving left 2 units then right 3 units cancels. A negative displacement would be moving left that many units.

(b) displacement $= S(b) \Rightarrow S(b) = \int_a^b s'(t)dt + S(a)$

$S(2) - S(0) = \int_0^2 s'(t)\,dt$

$S(2) - 2 = \triangle$

$S(2) - 2 = \frac{1}{2}(2)(-1)$

$S(2) - 2 = -1$

$S(2) = 1$