Week01: January 18th, 2023

Review

Basic Properties of Derivatives

Addition: $\frac{d}{dx}[f(x)+g(x)] = [f(x)+g(x)]' = f'(x)+g'(x) = \frac{d}{dx}f(x) + \frac{d}{dx}g$

Subtraction: $\frac{d}{dx}[f(x)-g(x)] = [f(x)-g(x)]' = f'(x)-g'(x) = \frac{d}{dx}f(x) - \frac{d}{dx}g$

Coefficient: $\frac{d}{dx}[c\cdot f(x)] = [c\cdot f(x)]' = c\cdot f'(x) = c\cdot\frac{d}{dx}f(x)$

Product: $\frac{d}{dx}[f(x)\cdot g(x)] = [f(x)\cdot g(x)]' = f'(x)\cdot g(x) + g'(x)\cdot f(x) = \frac{d}{dx}f(x)\cdot g(x) + \frac{d}{dx}g(x)\cdot f(x)$

Quotient: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2} = \frac{\frac{d}{dx}f(x)\cdot g(x) - \frac{d}{dx}g(x)\cdot f(x)}{(g(x))^2}$

Chain: $\frac{d}{dx}[f(g(x))] = [f(g(x))]' = f'(g(x))\cdot g'(x) = \frac{d}{dx}f(g(x))\cdot\frac{d}{dx}g(x)$

Common Derivatives

Constant: $\frac{d}{dx}(k) = 0$ k is a constant

Power: $\frac{d}{dx}(x^n) = nx^{n-1}$ n is a constant

Trig.: $\frac{d}{dx}(\sin(x)) = \cos(x)$

$\frac{d}{dx}(\cos(x)) = -\sin(x)$

$\frac{d}{dx}(\tan(x)) = \sec^2(x)$

$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$

$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$

$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$

Basic Integrals

Power: $\int x^n\,dx = \frac{1}{n+1}\cdot x^{n+1} + c$

Trig: $\int \sin(x)\,dx = -\cos(x) + c$

$\int \cos(x)\,dx = \sin(x) + c$

$\int \sec^2(x)\,dx = \tan(x) + c$

$\int \csc^2(x)\,dx = -\cot(x) + c$

$\int \csc(x)\cot(x)\,dx = -\csc(x) + c$

$\int \sec(x)\tan(x)\,dx = \sec(x) + c$

Derivative of Exponential & Logarithmic Functions

Logarithmic Properties

Product: $\ln(ab) = \ln(a) + \ln(b)$

Quotient: $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

Power: $\ln(a^n) = n\cdot\ln(a)$

Log of 1: $\ln(1) = 0$

ln of e: $\ln(e) = 1$

others: $\ln(e^x) = x\cdot\ln(e) = x$

$e^{\ln(x)} = x$

Exponential Rules

Product: $a^n\cdot a^m = a^{n+m}$

Quotient: $\frac{a^m}{a^n} = a^{n-m}$

Power: $a^n\cdot b^n = (ab)^n$

$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$

Derivatives

Exponential: $\frac{d}{dx}(b^x) = b^x\cdot\ln(b)$

$\frac{d}{dx}(e^x) = e^x\cdot\ln(e) = e^x$

Logarithmic: $\frac{d}{dx}(\log_b(x)) = \frac{1}{x\ln(b)}$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x\ln(e)} = \frac{1}{x}$

Anti-derivative

Exponential: $\int b^x\,dx = \frac{1}{\ln(b)}\cdot b^x + c$

$\int e^x\,dx = \frac{1}{\ln(e)}\cdot e^x + c = e^x + c$

Logarithmic: $\int \frac{1}{x\ln(b)}\,dx = \log_b(x) + c$

$\int \frac{1}{x}\,dx = \ln(x) + c$

Exercises

1. Find the following derivatives:

(a) $\frac{d}{dx}(\underbrace{x^3}_{f}\underbrace{\tan(x)}_{g})$

$(x^3)'\cdot\tan(x) + (\tan(x))'\cdot x^3$

$3x^2\tan(x) + \sec^2(x)\cdot x^3$

$\boxed{3x^2\tan(x) + x^3\sec^2(x)}$

product rule $f'g+g'f$

power rule nx^{n-1}

(b) $\frac{d}{dx}(\overbrace{\underbrace{\sqrt[3]{2x^2-5x+3}}_{g(x)}}^{f(x)})$

$(2x^2-5x+3)^{1/3}$

$((2x^2-5x+3)^{1/3})'\cdot(2x^2-5x+3)'$

$\frac{1}{3}(2x^2-5x+3)^{-2/3}\cdot(4x-5)$

$\boxed{\frac{4x-5}{3\sqrt[3]{2x^2-5x+3}}}$

chain rule $f'(g(x))\cdot g'(x)$

power rule nx^{n-1}

2. Find the equation of the line tangent to the curve $x\cos(1+2y)=2y^2-8$ at the point $(0,2)$.

Equation of the Tangent Line
- the derivative is the slope of the tangent line
- point-slope linear form
  $y-y_0=m(x-x_0)$

Implicit Differentation
- take derivative of x terms normally
- multiply by y' when taking the derivative of y with respect to x

$\underbrace{x}_{f}\overbrace{\cos(\underbrace{1+2y}_{h})}^{g}=2y^2-8$

$f'(x)\cdot g(h(y)) + (g'(h(y))\cdot h'(y))\cdot f(x)$

$1\cdot\cos(1+2y) - \sin(1+2y)\cdot 2y'\cdot x = 4yy'$

$\cos(1+2y) = 4yy' + 2x\sin(1+2y)\cdot y'$

$\cos(1+2y) = y'(4y+2x\sin(1+2y))$

$y'=\frac{\cos(1+2y)}{4y+2x\sin(1+2y)}$

@ $(0,2)$: $y'=\frac{\cos(1+2(2))}{4(2)+2(0)\sin(1+2(2))}$

$=\frac{\cos(5)}{8+0}$

$=\frac{1}{8}\cos(5)$

$y-2=\frac{1}{8}\cos(5)(x-0)$

$y-2=\frac{1}{8}\cos(5)x$

$\boxed{y=\frac{1}{8}\cos(5)x+2}$

3. Find a formula for the function $f(x)$ if its slope is given by $x\sin(x^2+1)$ and the graph of $f(x)$ passes through the point $(1,2)$.

! We can not assume f(x) is linear so we use initial value rather than point-slope linear form!

Initial Value Problem
- $f(x)=\int f'(x)\,dx$
- solve for c using initial value

$f(x)=\int x\sin(x^2+1)\,dx$

$u=x^2+1$ $\quad du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$=\int\sin(u)\cdot\frac{1}{2}du$

$=\frac{1}{2}\int\sin(u)\,du$

$=\frac{1}{2}(-\cos(u))+c$

$=-\frac{1}{2}\cos(u)+c$

$=-\frac{1}{2}\cos(x^2+1)+c$

$2=f(1)=-\frac{1}{2}\cos(5)+c$

$2+\frac{1}{2}\cos(5)=c$

U-Substitution
- identify u and its derivative du
- simply to receive direct substitution
- solve for new bounds by plugging in the original bounds into the u-equation

$\boxed{f(x)=-\frac{1}{2}\cos(x^2+1)+\frac{1}{2}\cos(5)+2}$

4. Evaluate $\int_0^1\frac{x^2+2}{\sqrt{x^3+6x+5}}\,dx$

U-Substitution
- identify u and its derivative du
- simply to receive direct substitution
- solve for new bounds by plugging in the original bounds into the u-equation

$\int_0^1(x^3+6x+5)^{-1/2}\cdot(x^2+2)\,dx$

$u=x^3+6x+5$

$du=(3x^2+6)\,dx$

$\frac{1}{3}du=(x^2+2)\,dx$

$=\int_5^{12}u^{-1/2}\cdot\frac{1}{3}du$

$=\frac{1}{3}\int_5^{12}u^{-1/2}\,du$

$=\frac{1}{3}\left[\frac{1}{-1/2+1}u^{-1/2+1}\right]_5^{12}$

bounds

lower $u=5$

upper $u=12$

$=\frac{1}{3}\left[\frac{1}{1/2}u^{1/2}\right]_5^{12}$

$=\frac{1}{3}\left[2u^{1/2}\right]_5^{12}$

$=\frac{1}{3}\left[2((12)^{1/2}-(5)^{1/2})\right]$

$=\frac{1}{3}\left[2(\sqrt{12}-\sqrt{5})\right]$

$=\frac{1}{3}\left[2(2\sqrt{3}-\sqrt{5})\right]$

$=\frac{2}{3}(2\sqrt{3}-\sqrt{5})$

$=\boxed{\frac{4}{3}\sqrt{3}-\frac{2}{3}\sqrt{5}}$