Exponential & Logarithmic Functions

Logarithmic Properties

Product: $\ln(ab) = \ln(a) + \ln(b)$

Quotient: $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

Power: $\ln(a^n) = n \cdot \ln(a)$

ln of 1: $\ln(1) = 0$

ln of e: $\ln(e) = 1$

others: $\ln(e^x) = x \cdot \ln(e) = x$

$\ln(ax^b) = \ln(a) + b \cdot \ln(x)$

Exponential Rules

Product: $a^n \cdot a^m = a^{n+m}$

Quotient: $\frac{a^m}{a^n} = a^{n-m}$

Power: $a^n \cdot b^n = (ab)^n$

$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$

e to ln: $e^{\ln(x)} = x$

Derivatives

Exponential: $\frac{d}{dx}(b^x) = b^x \cdot \ln(b)$

$\frac{d}{dx}(e^x) = e^x \cdot \ln(e) = e^x$

Logarithmic: $\frac{d}{dx}(\log_b(x)) = \frac{1}{x\ln(b)}$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x\ln(e)} = \frac{1}{x}$

Anti-derivative

Exponential: $\int b^x\,dx = \frac{1}{\ln(b)} \cdot b^x + c$

$\int e^x\,dx = e^x + c$

Logarithmic: $\int \frac{1}{x\ln(b)}\,dx = \log_b|x| + c$

$\int \frac{1}{x}\,dx = \ln|x| + c$

(Re-)Defining natural log:

1. Consider the area function $f(x) = \int_1^x \frac{1}{t}\,dt$ for $x > 0$. We call $f(x)$ the logarithm function and denote it by $f(x) = \ln x$.

a. $f'(x) = \frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] \overset{?}{=} \underline{\quad \frac{1}{x} \quad}$ $(x > 0)$

2nd FTC

b. $\frac{d}{dx}[\ln|x|] \overset{?}{=} \underline{\quad \frac{1}{x} \quad}$ $(x \neq 0)$

c. What can you say about ln(1)? Define the value of e using the definition of the natural logarithm.

$\ln(1) = f(1) = \int_1^1 \frac{1}{t}\,dt = 0$

Define e to be the number such that $\ln(e) = 1$.

d. Using the Fundamental Theorem of Calculus, show that $\ln(ax) = \ln(a) + \ln(x)$. Prove further that (ii) $\ln(e^n) = n$ where n is an integer and (iii) $\ln(e^r) = r$ where r us any rational number.

i. $\ln(ax) = \int_1^{ax} \frac{1}{t}\,dt$

$= \int_1^{a} \frac{1}{t}\,dt + \int_a^{ax} \frac{1}{t}\,dt$

$= \int_1^{a} \frac{1}{t}\,dt + \int_a^{ax} \frac{a}{t}\frac{1}{a}\,dt$ $\quad u = \frac{t}{a} \Rightarrow \frac{1}{u} = \frac{a}{t}$

$= \int_1^{a} \frac{1}{t}\,dt + \int_1^{x} \frac{1}{u}\,du$ $\quad du = \frac{1}{a}dt$

$= \ln(a) + \ln(x)$ $\quad a' = \frac{a}{a} = 1$

$b' = \frac{ax}{a} = x$

iii. $\ln(e^r) = r \cdot \ln(e)$

$= r \cdot 1$

$= r$

ii. $\frac{d}{dx}(\ln(x^n)) = \frac{1}{x^n} \cdot n \cdot x^{n-1} = \frac{n}{x}$

$\frac{d}{dx}(n \cdot \ln(x)) = n \cdot \frac{1}{x} = \frac{n}{x}$

$\int \frac{n}{x}\,dx = \ln(x^n) + c_1$

$\int \frac{n}{x}\,dx = n \cdot \ln(x) + c_2$ ← can't reuse c

$\ln(x^n) + c_1 = n \cdot \ln(x) + c_2$

$\ln(x^n) = n \cdot \ln(x) + (c_2 - c_1)$ ← $c_2 - c_1$ is just a constant

$= n \cdot \ln(x) + c$

$\ln(1^n) = n \cdot \ln(1) + c$

$0 = n \cdot 0 + c$

$0 = c$

$\ln(x^n) = n \cdot \ln(x) + 0$

Example A. Find the area under the graph of $y = \dfrac{-2}{4x-3}$ for $0 \le x \le 1/2$.

Area under function $f(x)$ from $a \le x \le b$ is equal to $\int_a^b f(x)dx$

$\int_0^{1/2} \frac{-2}{4x-3}\,dx$

$u = 4x - 3$ $\quad a' = 4(0) - 3 = -3$

$du = 4\,dx$ $\quad b' = 4(1/2) - 3$

$\frac{1}{2}du = 2dx$ $\quad = 2 - 3 = -1$

$\int_{-3}^{-1} \frac{-1}{u} \cdot \frac{1}{2}\,du$

$= -\frac{1}{2}\int_{-3}^{-1} \frac{1}{u}\,du$

$= -\frac{1}{2}[\ln|-1| - \ln|-3|]$

$= -\frac{1}{2}[\ln(1) - \ln(3)]$

$= -\frac{1}{2}[0 - \ln(3)]$

$= \frac{1}{2}\ln(3)$

In general, $\int \frac{1}{ax+b}\,dx$

$u = ax + b$

$du = a\,dx$

$\frac{1}{a}du = dx$

$\int \frac{1}{u} \cdot \frac{1}{a}\,du$

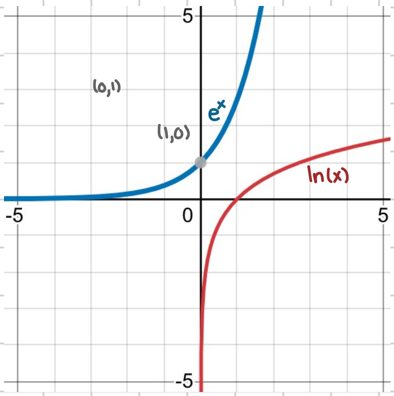
$= \frac{1}{a}\int \frac{1}{u}\,du$

$= \frac{1}{a}\ln|u| + c$

$= \frac{1}{a}\ln|ax+b| + c$

e. Give a sketch of the graph of $y = \ln x$. State clearly the domain and range of $\ln x$. What are the values of $\lim_{x \to 0^+} \ln x$ and $\lim_{x \to \infty} \ln x$?

f. The inverse $g(x)$ of $f(x) = \ln x$ exists. Why? Sketch the graph of $g(x) = \exp(x)$. Infer from (d) that we may write $\exp(x) = e^x$ for all real value x.



$y = \ln(x)$

domain: $0 < x < \infty$
↳ input OR $(0, \infty)$

range: $-\infty < y < \infty$
↳ output OR $(-\infty, \infty)$

$y = e^x$

domain: $(-\infty, \infty)$
OR $-\infty < x < \infty$

range: $(0, \infty)$
OR $0 < y < \infty$

Notice:
$\ln(1) = 0$
$e^0 = 1$

h. Using the fact that $\frac{d}{dx}(e^x) = e^x$, the chain rule and the fact that $e^{\ln b} = b$ $(b > 0)$, show that $\frac{d}{dx}(b^x) = b^x \ln b$.

$$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{\ln(b^x)}\right) = \frac{d}{dx}\left(e^{x\overbrace{\ln(b)}^{\text{constant}}}\right) = e^{x \cdot \ln(b)} \cdot \ln(b)$$

i. Using the change of base formula $\log_b x = \frac{\ln x}{\ln b}$, show that $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$.

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln(x)}{\ln(b)}\right) = \frac{d}{dx}\left(\overbrace{\frac{1}{\ln(b)}}^{\text{constant}} \cdot \ln(x)\right) = \frac{1}{\ln(b)} \cdot \frac{1}{x} = \frac{1}{x\ln(b)}$$

Example B. Find the equation of the tangent line to the curve $y = 4 - 2e^x + \ln\left(\frac{1 - x^2}{1 + x^2}\right)$ at $x = 0$.

Equation of the tangent line:
line: $y - y_1 = m(x - x_1)$
slope = derivative: $m = \frac{dy}{dx}\Big|_{x_1} = y'(x_1)$

Using log rules in $\frac{d}{dx}$:

$\ln\left(\frac{f(x)}{g(x)}\right) = \ln(f(x)) - \ln(g(x))$
chain + quotient $\quad$ chain + chain

Given $x_1 = 0$, $y_1 = 4 - 2e^0 + \ln\left(\frac{1+0^2}{1+0^2}\right)$
$= 4 - 2 \cdot 1 + \ln(1)$
$= 4 - 2 + 0 = 2$

$y = 4 - 2e^x + \ln\left(\frac{1-x^2}{1+x^2}\right)$
$y = 4 - 2e^x + \ln(1-x^2) - \ln(1+x^2)$
$y' = 0 - 2e^x + \frac{1}{1-x^2} \cdot (-2x) - \frac{1}{1+x^2}(2x)$

$y'(0) = -2e^0 + 0 - 0 = -2 \cdot 1 = -2$

$y - 2 = -2(x - 0)$
$y - 2 = -2x$
$y = -2x + 2$

Exit Ticket Derivative and Integral Review

Fill in the derivatives and integrals:

1. $\frac{d}{dx}[k] = 0$
2. $\int k dx = kx + C$
3. $\frac{d}{dx}[kx^n] = k \cdot n \cdot x^{n-1}$
4. $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$
5. $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$
6. $\int \frac{1}{x} dx = \ln|x| + C$
7. $\frac{d}{dx}[\log_a(x)] = \frac{1}{x \cdot \ln(a)}$
8. $\int \frac{1}{x \cdot \ln(a)} dx = \log_a(x) + C$
9. $\frac{d}{dx}[e^x] = e^x$
10. $\int e^x dx = e^x + C$
11. $\frac{d}{dx}[a^x] = a^x \cdot \ln(a)$
12. $\int a^x dx = \frac{1}{\ln(a)} a^x + C$
13. $\frac{d}{dx}[\sin(x)] = \cos(x)$
14. $\int \cos(x) dx = \sin(x) + C$
15. $\frac{d}{dx}[\cos(x)] = -\sin(x)$
16. $\int \sin(x) dx = -\cos(x) + C$
17. $\frac{d}{dx}[\tan(x)] = \sec^2(x)$
18. $\int \sec^2(x) dx = \tan(x) + C$
19. $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
20. $\int \sec(x)\tan(x) dx = \sec(x) + C$

Use the rules above to find the integrals below and check your answer:

1. $\int \cot(x)\sin(x)dx$

$= \int \frac{\cos(x)}{\sin(x)} \cdot \sin(x)\,dx$

$= \int \cos(x)dx$

$= \sin(x) + C$

2. $\int \frac{1+\cos^2(\theta)}{\cos^2(\theta)} d\theta$

$= \int \frac{1}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} d\theta$

$= \int \sec^2(\theta) + 1\, d\theta$

$= \tan(\theta) + \theta + C$

3. $\int \frac{2u^2 - 5u + \sqrt[3]{u}}{u^2} du$

$= \int \frac{2u^2}{u^2} - \frac{5u}{u^2} + \frac{u^{1/3}}{u^2} du$

$= \int 2 - 5\frac{1}{u} + u^{-5/3} du$

$= 2u - 5\ln|u| - \frac{3}{2} u^{-2/3} + C$

4. $\int 6x(x^2+1)^2 dx$

$u = x^2 + 1$

$du = 2x\, dx$

$= \int 3u^2 du$

$= u^3 + C$

$= (x^2+1)^3 + C$