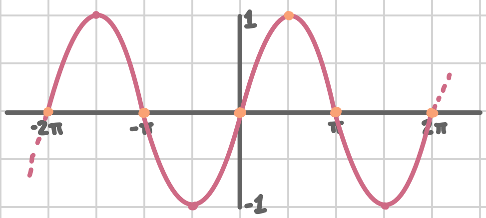
Inverse Trigonometric Functions

Graphical Representation

In its simplest form, the inverse of the function $f(x)$ that send x to y is just the function $g(y)$ that sends y to x for all x in the domain. In words, if you apply f then g to a number x, $g(f(x))$, then you get x back. We can use this knowledge to graph the inverses of trigonometric functions

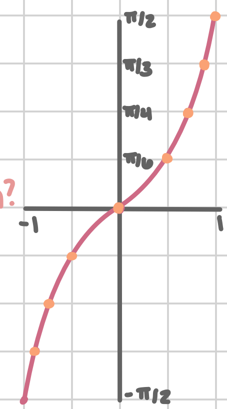
$y = \sin(x)$



Key points:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$-\frac{\pi}{6}$	$-\frac{\pi}{4}$	$-\frac{\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{2\pi}{3}$
y	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	wait! hmm...	$-\frac{\pi}{6}$	$-\frac{\pi}{4}$	$-\frac{\pi}{3}$	$-\frac{\pi}{2}$	again?

$y = \arcsin(x)$



We ran into an issue... we want to say $\frac{\sqrt{3}}{2}$ goes to $\frac{2\pi}{3}$, but we have already said it goes to $\frac{\pi}{3}$. It would fail the vertical line tes if we sent it to both so we must stop here.

Now let us find domain and range from these graphs:

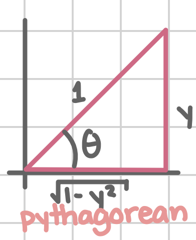
(i) $y = \sin(x)$ domain: $(-\infty, \infty)$ range: $[-1, 1]$

(ii) $y = \arcsin(x)$ domain: $[-1, 1]$ can you see how these relate? range: $[-\pi/2, \pi/2]$

Trigonometric Representation

Now that was great visually, but could you tell me derivative of $\arcsin(x)$? Probably not from this alone. So we go back to triangles - math's best friend.

Let us recall geometrically what sine means and then infer its inverse. We start with a triangle with hypotenuse of length 1.



$\sin(\theta) = \frac{y}{1}$

takes an angle — gives a ratio

$\arcsin(y) = \theta$

takes a ratio — gives an angle

Algebraically: $\sin(\theta) = y$

$\arcsin(\sin(\theta)) = \arcsin(y)$

$\theta = \arcsin(y)$

Derivatives & Integrals

We can now use this geometric definition to find the derivative of $\theta = \arcsin(y)$ using implicit differentation and the triangle above.

$$\theta = \arcsin(y) \longleftarrow \frac{d\theta}{dy} = \frac{d}{dy}[\arcsin(y)]$$ which is unknown

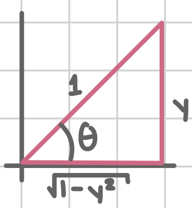
$$\sin(\theta) = y$$ established above

$$\frac{d}{dy}[\sin(\theta)] = \frac{d}{dy}[y]$$ derive with respect to variable needed

$$\cos(\theta) \cdot \frac{d\theta}{dy} = 1$$

$$\frac{d\theta}{dy} = \frac{1}{\cos(\theta)}$$

We almost have what we want. There is a pesky $\cos(\theta)$ that we need to replace using the triangle.



$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{\sqrt{1-y^2}}{1}$$

Thus $\frac{d\theta}{dy} = \frac{1}{\cos(\theta)} = \frac{1}{\sqrt{1-y^2}}$.

The rest of the inverse trigonometric functions can be derived similiarly.

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + c$$

$$\frac{d}{dx}[\arccos(x)] = \frac{-1}{\sqrt{1-x^2}} \qquad \int \frac{-1}{\sqrt{1-x^2}}\,dx = -\arcsin(x) + c \longleftarrow$$ we use the integral rule $\int c \cdot f(x)dx = c \cdot \int f(x)dx$

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \arctan(x) + c$$

Examples:

1. Find the derivative of $y = (1+2x)^{\arctan(x)}$.

We do not know how to take the derivative of $f(x)^{g(x)}$ so we use log rules.

$$\ln(y) = \ln[(1+2x)^{\arctan(x)}]$$ ln both sides

$$\ln(y) = \underbrace{\arctan(x) \cdot \ln(1+2x)}_{\text{product rule: } f'g + g'f}$$ $\ln(x^a) = a \cdot \ln(x)$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \left(\frac{1}{1+x^2}\right)(\ln(1+2x)) + \left(\frac{1}{1+2x}\cdot 2\right)(\arctan(x))$$

$$\frac{dy}{dx} = y\cdot\left[\frac{\ln(1+2x)}{1+x^2} + \frac{2}{1+2x}\cdot\arctan(x)\right]$$

$$\frac{dy}{dx} = (1+2x)^{\arctan(x)}\cdot\left[\frac{\ln(1+2x)}{1+x^2} + \frac{2}{1+2x}\cdot\arctan(x)\right]$$

2. Find the derivative of $\arcsin(2x+y^2)$ with respect to x while treating y as a constant.

Recall $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$, thus $\frac{d}{dx}[\arcsin(u)] = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$

$$\frac{d}{dx}[\arcsin(2x+y^2)] = \frac{1}{\sqrt{1-(2x+y^2)^2}}\cdot(2)$$

y is a constant so $\frac{d}{dx}[2x+y^2] = 2$

3. Find the derivative of $\arcsin(2x+y^2)$ with respect to y while treating x as a constant.

$$\frac{d}{dy}[\arcsin(2x+y^2)] = \frac{1}{\sqrt{1-(2x+y^2)^2}}\cdot(2y)$$

x is a constant so $\frac{d}{dy}[2x+y^2] = 2y$

4. A population $y(t)$ (in units of millions) of bacteria grows according to the rate $\frac{dy}{dt} = \frac{1}{1+4t^2}$. Find the total change in the size of the population over the time duration $0 \le t \le 1/2$.

definite integral

$$\int_0^{1/2} \frac{1}{1+4t^2}\,dt$$

so close to $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

$$\int_0^{1/2} \frac{1}{1+(2t)^2}\,dt$$

$4t^2 = 2^2\cdot t^2 = (2t)^2$ so we have $1+u^2$ now

$u = 2t$ $\quad du = 2dt$ $\quad$ lower $= 2(0) = 0$ $\quad$ upper $= 2(1/2) = 1$

$$\int_0^1 \frac{1}{1+u^2}\cdot\frac{1}{2}\,du$$
$$= \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du$$
$$= \frac{1}{2}\arctan(u)\Big|_0^1$$
$$= \frac{1}{2}[\arctan(1) - \arctan(0)]$$
$$= \frac{1}{2}[\pi/4 - 0]$$
$$= \pi/8$$

$\arctan(1) = \theta$
$1 = \tan(\theta)$
$\theta = \pi/4$

$\arctan(0) = \theta$
$0 = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
$0 = \sin(\theta)$
$\theta = 0$

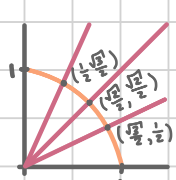
5. Use the unit circle or triangles to find:

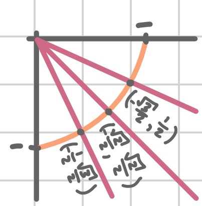
(a) $\arcsin(\sqrt{3}/2)$
$= \pi/3$

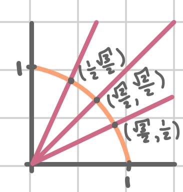
(b) $\arcsin(-\sqrt{3}/2)$
$= -\frac{\pi}{3}$

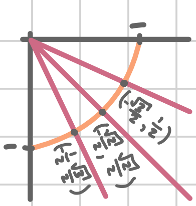
c. $\arccos(\frac{1}{2})$
$= \pi/3$

d. $\arctan(-1)$
$= -\pi/4$







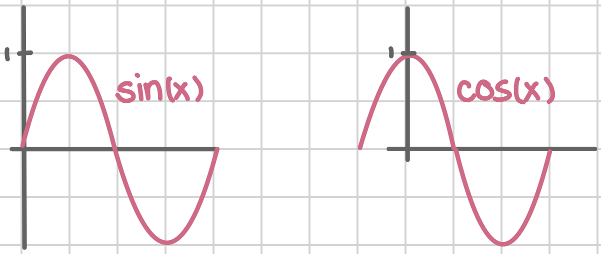


How to build a unit circle:

Previous knowledge:

$\sin(0) = 0$
$\cos(0) = 1$

I always think of their graphs:

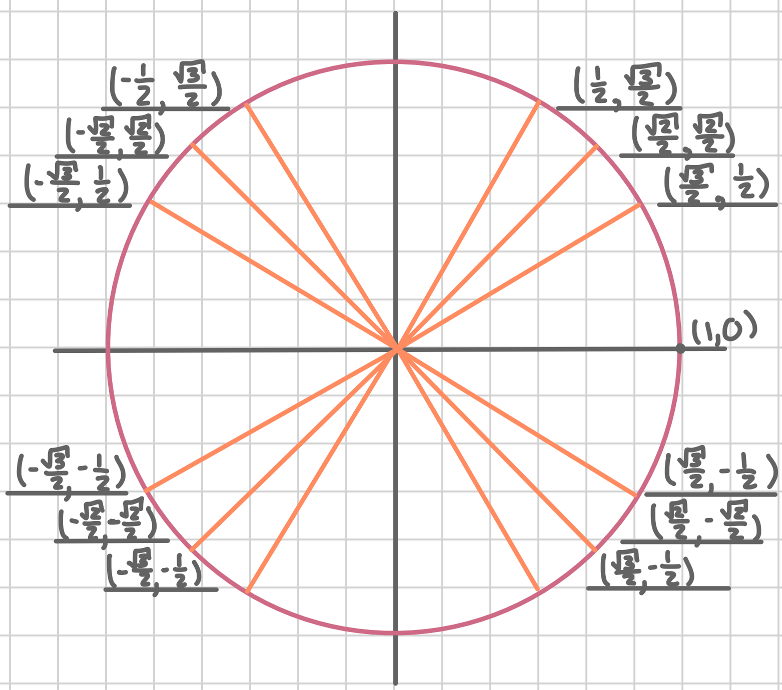


First Quadrant:

radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin(x)	$\sqrt{0/4}$	$\sqrt{1/4}$	$\sqrt{2/4}$	$\sqrt{3/4}$	$\sqrt{4/4}$
sin(x)	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos(x)	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan(x)	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und.

Step 1: count 0-4
Step 2: divide by 4
Step 3: square root
Step 4: simplify
Step 5: reverse order
Step 6: divide $\sin(x)/\cos(x)$

Draw Circle:



Quadrant 1:
Use chart above
$(\cos(\theta), \sin(\theta))$

I know this as $\sin(0)=0$ so it must be the y

Quadrant 2:
Flip over y-axis
$(x, y) \rightarrow (-x, y)$

Quadrant 3:
Flip over x-axis
$(-x, y) \rightarrow (-x, -y)$

like folding paper

Quadrant 4:
Flip over y-axis
$(-x, -y) \rightarrow (x, -y)$

Exit Ticket Natural Log

Fill in the following rules:

1. $\ln(a) + \ln(b) =$ $\ln(ab)$
2. $\ln(a) - \ln(b) =$ $\ln(\frac{a}{b})$
3. $\ln(x^a) =$ $a \cdot \ln(x)$
4. $\ln(ax^b) =$ $\ln(a) + b\ln(x)$
5. $\frac{d}{dx}[\ln(ax+b)] =$ $\frac{1}{ax+b} \cdot a$
6. $\int \frac{a}{ax+b}dx =$ $\ln|ax+b| + C$

Use the above rules to solve the following equations for x:

1. $\int \frac{1}{2x+5}dx$ $u = 2x+5$

$= \frac{1}{2}\ln|2x+5| + C$

2. $\int \frac{1}{x+12}dx$

$= \ln|x+12| + C$

3. $\frac{d}{dx}\left[\ln\left(\frac{1-x}{1+x}\right)\right]$

$= \frac{d}{dx}[\ln(1-x) - \ln(1+x)]$

$= \frac{-1}{1-x} - \frac{1}{1+x}$

4. $\frac{d}{dx}\left[\ln\left(\frac{2x^2-3}{3x^3-6}\right)\right]$

$= \frac{d}{dx}[\ln(2x^2-3) + \ln(3x^3-6)]$

$= \frac{4x}{2x^2-3} + \frac{9x^2}{3x^2-6}$

5. $\int \frac{2x}{4x^2+12}dx$

$u = 4x^2 + 12$

$du = 8x\,dx$

$\frac{1}{4}du = 2x\,dx$

$= \int \frac{1}{u} \cdot \frac{1}{4}du$

$= \frac{1}{4}\ln|u| + C$

$= \frac{1}{4}\ln|4x^2+12| + C$

$= \frac{1}{4}\ln(4x^2+12) + C$

6. $\int \frac{5x+7}{5x^2+14x+6}dx$

$u = 5x^2 + 14x + 6$

$du = (10x + 14)dx$

$du = 2(5x+7)dx$

$\frac{1}{2}du = (5x+7)\,dx$

$= \int \frac{1}{u} \cdot \frac{1}{2}du$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|5x^2+14x+6| + C$

$= \frac{1}{2}\ln(5x^2+14x+6) + C$