Week02: January 26th, 2023

Review: Logarithmic Functions

Logarithmic Properties

Product: $\ln(ab) = \ln(a) + \ln(b)$

Quotient: $\ln(\frac{a}{b}) = \ln(a) - \ln(b)$

Power: $\ln(a^n) = n \cdot \ln(a)$

Log of 1: $\ln(1) = 0$

ln of e: $\ln(e) = 1$

others: $\ln(e^x) = x \cdot \ln(e) = x$

$e^{\ln(x)} = x$

Exponential Rules

Product: $a^n \cdot a^m = a^{n+m}$

Quotient: $\frac{a^m}{a^n} = a^{n-m}$

Power: $a^n \cdot b^n = (ab)^n$

$\frac{a^n}{b^n} = (\frac{a}{b})^n$

Change of Base

$\log_b x = \frac{\ln x}{\ln b}$

Derivatives

Exponential: $\frac{d}{dx}(b^x) = b^x \cdot \ln(b)$

$\frac{d}{dx}(e^x) = e^x \cdot \ln(e) = e^x$

Logarithmic: $\frac{d}{dx}(\log_b(x)) = \frac{1}{x\ln(b)}$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x\ln(e)} = \frac{1}{x}$

Anti-derivative

Exponential: $\int b^x dx = \frac{1}{\ln(b)} \cdot b^x + c$

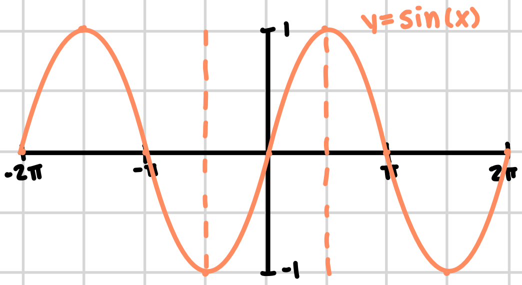
$\int e^x dx = \frac{1}{\ln(e)} \cdot e^x + c = e^x + c$

Logarithmic: $\int \frac{1}{x\ln(b)} dx = \log_b(x) + c$

$\int \frac{1}{x} dx = \ln(x) + c$

Inverse Trigonometric Functions

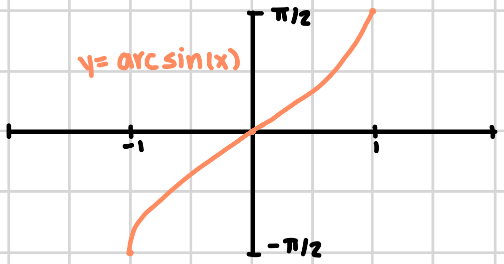
$\sin^{-1}(x) = \arcsin(x)$



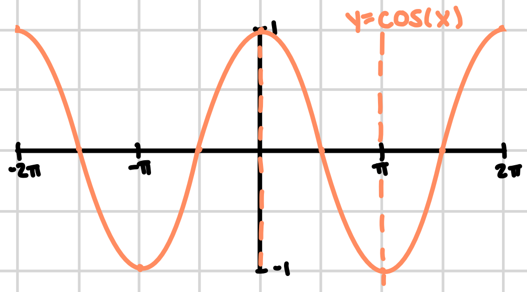
$\arcsin(0) = 0$

$\arcsin(1) = \pi/2$

$\arcsin(-1) = -\pi/2$



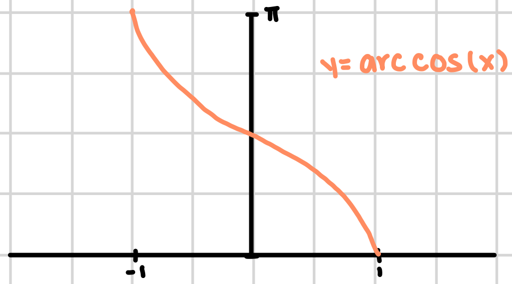
$\cos^{-1}(x) = \arccos(x)$



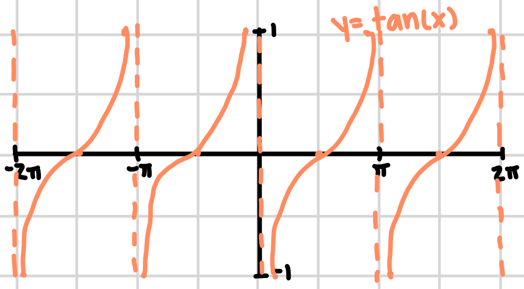
$\arccos(0) = \pi/2$

$\arccos(1) = 0$

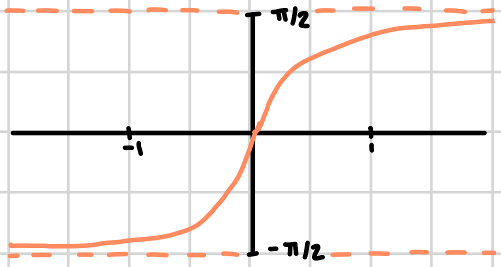
$\arccos(-1) = \pi$



$\tan^{-1}(x) = \arctan(x)$



$\arctan(0) = \pi/2 + k\pi$



Derivatives

Inverse Trig.: $\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\arccos(x)) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

Anti-Derivative

Inverse Trig.: $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x)$

$\int \frac{1}{1+x^2} dx = \arctan(x)$

More Examples:

1. Using the log function, find the derivative of $y = (1+2x)^{\arctan(x)}$

$\frac{d}{dx}((1+2x)^{\arctan(x)}) = \frac{d}{dx}(e^{\ln((1+2x)^{\arctan(x)})}) = \frac{d}{dx}(e^{(\arctan(x))\ln(1+2x)}) = e^{\arctan(x)\ln(1+2x)} \cdot (\frac{1}{1+x^2} \cdot \ln(1+2x) + \arctan(x) \cdot \frac{2}{1+2x})$

$= e^{\ln(1+2x)^{\arctan(x)}} \cdot (\frac{1}{1+x^2} \cdot \ln(1+2x) + \arctan(x) \cdot \frac{2}{1+2x}) = (1+2x)^{\arctan(x)} \cdot (\frac{1}{1+x^2} \cdot \ln(1+2x) + \arctan(x) \cdot \frac{2}{1+2x})$

2. Find the derivative of $\arcsin(2x+y^2)$ with respect to x treating y as a constant.

$\frac{d}{dy}(\arcsin(2x+y^2)) = \frac{1}{\sqrt{1-(2x+y^2)^2}} \cdot (2)$ when you treat y as a constant imagine it as a number, $\frac{d}{dx}(xy^2) = y^2$, $\frac{d}{dx}(y^n) = 0$

3. Find the derivative of $\arcsin(2x+y^2)$ with respect to y treating x as a constant.

$\frac{d}{dy}(\arcsin(2x+y^2)) = \frac{1}{\sqrt{1-(2x+y^2)^2}} \cdot (2y)$ this may seem werid at first but it is the same as 2 with the variables exchanged

Methods of Substitution

Important Integral Formulas

$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c \quad \text{if } n \neq -1$

$\int \frac{1}{x} \, dx = \int x^{-1} dx = \ln|x| + c$

$\int \frac{1}{ax+b} \, dx = \frac{\ln(ax+b)}{a} + c$

$\int e^{ax+b} \, dx = \frac{e^{ax+b}}{a} + c$

$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x) + c$

$\int \frac{1}{1+x} \, dx = \arctan(x) + c$

Examples.

1. If the slope at each point of the graph of f(x) is given by $\frac{2x+1}{4+x^2}$. Find a formula for f(x) if its graph passes through (2,0).

$f(x) = \int \frac{2x+1}{4+x^2} \, dx$

$= \underset{①}{\int \frac{2x}{4+x^2} dx} + \underset{②}{\int \frac{1}{4+x^2} dx}$

① $\int \frac{2x}{4+x^2} dx$

$u = 4+x^2$

$du = 2x dx$

$\int \frac{1}{u} \, du$

$= \ln|u| + c_1$

$= \ln|4+x^2| + c_1$

② $\int \frac{1}{4+x^2} dx$

$= \frac{1}{4} \int \frac{1}{1+\frac{1}{4}x^2} dx$

$= \frac{1}{4} \int \frac{1}{1+(x/2)^2} dx$

$u = x/2$

$du = \frac{1}{2} dx \Rightarrow 2du = dx$

$= \frac{1}{4} \int \frac{1}{1+u^2} \cdot 2du$

$= \frac{1}{2} \arctan(u) + c_2$

$= \frac{1}{2} \arctan(x/2) + c_2$

$f(x) = ① + ② = \ln|4+x^2| + c_1 + \frac{1}{2}\arctan(x/2) + c_2$

$= \ln|4+x^2| + \frac{1}{2}\arctan(x/2) + c \qquad c = c_1 + c_2$

$0 = f(2) = \ln|4+(2)^2| + \frac{1}{2}\arctan(2/2) + c$

$0 = \ln|8| + \frac{1}{2} \cdot \frac{\pi}{4} + c$

$c = -\ln(8) - \frac{\pi}{8}$

$f(x) = \ln|4+x^2| + \frac{1}{2}\arctan(x/2) - \ln(8) - \frac{\pi}{8}$

Initial Value Problem

- $y = \int y'(x) \, dx$
- solve for c using initial value

2. Set up but do not solve the integral u-substitution.

(a) $\int \frac{x+2x^3}{1+x^2+x^4} dx$

$u = 1+x^2+x^4$

$du = (2x+4x^3)dx \Rightarrow \frac{1}{2} du = (x+2x^3) dx$

$= \int \frac{1}{u} \cdot \frac{1}{2} du$

(b) $\int \frac{1}{\sqrt{4-9x^2}} dx$

$= \int \frac{1/2}{\sqrt{1/4}} \cdot \frac{1}{\sqrt{4-9x^2}} dx$

$= \int \frac{1/2}{\sqrt{(1/4)\cdot(4-9x^2)}} dx$

$= \frac{1}{2} \int \frac{1}{\sqrt{1-9/4\, x^2}} dx$

$= \frac{1}{2} \int \frac{1}{\sqrt{1-(3x/2)^2}} dx$

$u = \frac{3}{2} x$

$du = 3/2 \, dx \Rightarrow 2/3 \, du = dx$

$= \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}} \cdot \frac{2}{3} du$

(c) $\int \frac{4+x}{\sqrt{1-9x^2}} dx$

$= \int \frac{4}{\sqrt{1-9x^2}} dx + \int \frac{x}{\sqrt{1-9x^2}} dx$

$= 4 \int \frac{1}{\sqrt{1-(3x)^2}} dx + \int \frac{x}{\sqrt{1-9x^2}} dx$

$u = 3x$

$du = 3dx \Rightarrow \frac{1}{3} du = dx$

$v = 1-9x^2$

$dv = -18x \, dx \Rightarrow -\frac{1}{18} dv = x dx$

$= 4 \int \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{3} du + \int \frac{1}{\sqrt{v}} \cdot -\frac{1}{18} dv$

(d) $\int \frac{e^{2t}}{1+e^{2t}} dt$

$u = 1+e^{2t}$

$du = e^{2t} \cdot 2 \, dt$

$\Rightarrow \frac{1}{2} du = e^{2t} dt$

$\int \frac{1}{u} \cdot \frac{1}{2} du$

(e) $\int \frac{e^t}{1+e^{2t}} dt$

$= \int \frac{e^t}{1+(e^t)^2} dt$

$u = e^t$

$du = e^t dt$

$= \int \frac{1}{1+u^2} du$

Exponential Growth and Decay

Many quantities (e.g. population) in the world can be modeled by the exponential growth/decay equation:

$$P(t) = P_0 e^{kt}$$

initial value, t=0

determines rate of change

If k is positive then we have an exponential growth function with growth constant k.

If k is negative then we have an exponential decay function with decay constant k.

P_0 is the initial size since $P(0) = P_0 e^0 = P_0 \cdot 1 = P_0$

doubling time. If k is positive then the doubling time is given by T s.t. $P(T) = 2P_0$.

half life. If k is negative then the half life is given by T s.t. $P(T) = \frac{1}{2} P_0$.

isotype. An isotope is a version of an element with the same number of protons but a different number nuetrons leading to a different atomic mass.

radioactive isotype. A radioactive isotope is an unstable nucleus that decomposes spontaneously.

double-time: $2P_0 = P(T) = P_0 e^{rT}$

$2 = e^{rT}$

$\ln(2) = rT$

$T = \frac{\ln(2)}{r}$

half-life: $\frac{1}{2} P_0 = P(T) = P_0 e^{rT}$

$\frac{1}{2} = e^{rT}$

$\ln(2^{-1}) = rT$

$T = \frac{-\ln(2)}{r}$

example. The population $P(t)$, at time t hours, of a bacteria is given by $P(t)=5e^{2t}$ in thousands.

(a) What is the initial population of the bacteria?

(b) Give a formula for the growth rate of the population of the bacteria.

(c) What did you observe about the growth rate.

(d) Explain what is meant by the doubling time for the population. Find this time.

(a) initial means $t=0$, $P(0)=5e^{2(0)}=5e^{0}=5\cdot 1=5$

(b) growth rate means derivative, $P'(t)=5e^{2t}\cdot 2=10e^{2t}$

(c) Notice that $P'(t)=P(t)\cdot 2$

(d) The doubling time of a quantity growing exponentially is the time needed for the population to double its initial amount, i.e. $q(T)=2q_0$

for half-time use $q(T)=\frac{1}{2}q_0$

If $P(t)=5e^{2t}$ then $P_0=5$ so when does $P(T)=2\cdot P_0=10$

$10=P(T)=5e^{2T}$

$2=e^{2T}$

$\ln(2)=\ln(e^{2T})$

$\ln(2)=2T$

$T=\frac{\ln(2)}{2}$

example. Recent expirements on viability of the coronavirus indicates that it reduces exponentially on various surfaces. The half life of the coronavirus on glass is estimated to be about 14 hours.

(a) Starting with 100% initially, find a formula in the form $A\cdot e^{rt}$ for the percentage of the virus on glass after t hours.

(b) If we consider the virus no longer infectious (or viable) after it is reduced to 1% or less, estimate how long will the virus remain infectious on glass.

(a) $q(t)=Ae^{rt}$

$A=100$

$r=?$

half life $=\frac{-\ln 2}{r}$

$14=-\frac{\ln 2}{r}$

$r=-\frac{\ln 2}{14}$

$q(t)=100e^{-\frac{\ln 2}{14}t}$

(b) We want to find T s.t. $q(T)=1$

$q(T)=100e^{-\frac{\ln 2}{14}t}=1$

$e^{-\frac{\ln 2}{14}T}=1/100$

$\ln(e^{-\frac{\ln 2}{14}T})=\ln(1/100)$

$-\frac{\ln 2}{14}T=\ln(100^{-1})=-\ln(100)$

$T=-\ln(100)\cdot\frac{-14}{\ln 2}$

$T=14\cdot\frac{\ln(100)}{\ln 2}\approx 93$ hours

example. A cypress beam found in the tomb of Sneferu in Egypt contained 55% of the amount of Carbon-14 found in living cypress wood. Estimate the age of the tomb given that Carbon-14 has a half-life of 5730 years.

We are given everything needed to set up the equation $q(t)=p_0e^{rt}$ then we need to solve for T s.t. $q(T)=55$

$q(t)=q_0e^{rt}$, $q_0=100\%$, $r=?$, $q(T)=55$

$r=\frac{-\ln 2}{5730}$

$q(t)=100e^{-(\ln 2/5730)t}$

$q(T)=100e^{-(\ln 2/5730)T}=55$

$e^{-(\ln 2/5730)T}=\frac{55}{100}$

$\ln(e^{-(\ln 2/5730)T}=\ln(55/100)$

$-(\ln 2/5730)T=\ln(11/20)$

$\frac{-\ln 2}{5730}T=\ln(11)-\ln(20)$

$T=-5730\cdot\frac{\ln(11)-\ln(20)}{\ln 2}$

$T=4{,}942.1$