Review of Week 1

Integral and Derivative Formulas

logarithmic: $\frac{d}{dx}[\ln|ax+b|] = \frac{1}{ax+b}\cdot(a)$ $\qquad \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

exponential: $\frac{d}{dx}[e^{ax+b}] = e^{ax+b}\cdot a$ $\qquad \int e^{ax+b}\,dx = \frac{1}{a}\cdot e^{ax+b} + c$

triganometric: $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$ $\qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + c$

$\frac{d}{dx}[\arccos(x)] = \frac{-1}{\sqrt{1-x^2}}$ $\qquad \int \frac{-1}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{\sqrt{1-x^2}}\,dx = -\arcsin(x) + c$

$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$ $\qquad \int \frac{1}{1+x^2}\,dx = \arctan(x) + c$

Examples

1. If the slope at each point of the graph of $f(x)$ is given by $\frac{2x+1}{4+x^2}$. Find a formula for $f(x)$ if its graph passes through $(2,0)$.

Initial Value Problem:
- $f(x) = \int f'(x)\,dx$
- solve for c

u-Substitution:
- u is "inner" function
- find du in integral

Inverse Trig. Integrals:
- use u-substitution to mimic $\int \frac{1}{1+u^2}\,du$ or $\int \frac{1}{\sqrt{1-u^2}}\,du$

$$f(x) = \int \frac{2x+1}{4+x^2}\,dx$$

first thought $u = 4+x^2$, but then $du = 2x$ no $+1$
separate the fraction

$$\int \frac{2x}{4+x^2} + \frac{1}{4+x^2}\,dx = \int \frac{2x}{4+x^2}\,dx + \int \frac{1}{4+x^2}\,dx$$

$$\int \frac{2x}{4+x^2}\,dx$$

$u = 4+x^2$
$du = 2x\,dx$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + c$$

$$= \ln|4+x^2| + c$$

$$\int \frac{1}{4+x^2}\,dx = \int \frac{1}{4(1+\frac{1}{4}x^2)}\,dx$$

factor out to make this 1

$$= \frac{1}{4}\int \frac{1}{1+(\frac{1}{2}x)^2}\,dx$$

$u = \frac{1}{2}x$
$du = \frac{1}{2}\,dx \Rightarrow 2du = dx$

$$= \frac{1}{4}\int \frac{1}{1+u^2}\cdot 2\,du$$

$$= \frac{1}{2}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\arctan(u) + c$$

$$= \frac{1}{2}\arctan\left(\frac{1}{2}x\right) + c$$

$f(x) = \ln|4+x^2| + \frac{1}{2}\arctan(\frac{1}{2}x) + c$

check: $f'(x) = \frac{2x}{4+x^2} + \frac{1}{2} \cdot \frac{1}{1+(\frac{1}{2}x)^2} \cdot \frac{1}{2}$ ✓

f(x) passes through (2,0)
thus $f(2)=0$

$f(2) = \ln|4+(2)^2| + \frac{1}{2}\arctan(\frac{1}{2}(2)) + c$

$0 = \ln|8| + \frac{1}{2}\arctan(1) + c$

$0 = \ln(8) + \frac{1}{2}(\pi/4) + c$

$-\ln(8) - \frac{\pi}{8} = c$

$f(x) = \ln|4+x^2| + \frac{1}{2}\arctan(\frac{1}{2}x) - \ln(8) - \frac{\pi}{8}$

2. Perform the following integrals:

(a) $\int_0^1 \frac{x+2x^3}{1+x^2+x^4}\,dx$

$u = 1+x^2+x^4$

$du = (2x+4x^3)dx$

$\int_1^3 \frac{1}{u} \cdot \frac{1}{2}\,du$

$= \frac{1}{2}\ln|u| \Big|_1^3$

$= \frac{1}{2}[\ln(3) - \ln(1)]$

$= \frac{1}{2}\ln(3)$

(b) $\int \frac{1}{\sqrt{4-9x^2}}\,dx$

$\int \frac{1}{2\sqrt{1-\frac{9}{4}x^2}}\,dx$

$\int \frac{1}{2\sqrt{1-(\frac{3}{2}x)^2}}\,dx$

$u = \frac{3}{2}x$

$du = \frac{3}{2}\,dx$

$\int \frac{1}{2\sqrt{1-u^2}} \cdot \frac{2}{3}\,du$

$= \frac{1}{3}\arcsin(u) + c$

$f(x) = \frac{1}{3}\arcsin(\frac{3}{2}x) + c$

$f'(x) = \frac{1}{3}\frac{1}{\sqrt{1+(3x/2)^2}} \cdot \frac{3}{2}$

(c) $\int \frac{4+x}{\sqrt{1-9x^2}}\,dx$

$\int \frac{4}{\sqrt{1-9x^2}}\,dx + \int \frac{x}{\sqrt{1-9x^2}}\,dx$

$\int \frac{4}{\sqrt{1-(3x)^2}}\,dx +$ $\quad u = 1-9x^2$, $du = -18x$

$v = 3x$, $dv = 3\,dx$ $\quad + \int \frac{1}{-18\sqrt{u}}\,du$

$\int \frac{4}{3\sqrt{1-v^2}}\,dv + \int -\frac{1}{18}u^{-1/2}\,du$

$= \frac{4}{3}\arcsin(v) - \frac{1}{18} \cdot 2\,u^{1/2} + c$

$= \frac{4}{3}\arcsin(3x) - \frac{1}{9}\sqrt{1-9x^2} + c$

$= \frac{4}{3}\frac{1}{\sqrt{1+(3x)^2}} \cdot 3 - \frac{1}{9} \cdot \frac{1}{2}(1-9x^2)^{-1/2} \cdot -18x$

(d) $\int_0^{\ln(2)} \frac{e^t}{1+e^{2t}}\,dt$

$\int_0^{\ln(2)} \frac{e^t}{1+(e^t)^2}\,dt$

$u = e^t$

$du = e^t\,dt$

$\int_1^2 \frac{1}{1+u^2}\,du$

$= \arctan(u) \Big|_1^2$

$= \arctan(2) - \arctan(1)$

(e) $\int \frac{e^{2t}}{1+e^{2t}}\,dt$

$u = 1+e^{2t}$

$du = e^{2t}dt$

$\int \frac{1}{u}\,du$

$= \ln|u| + c$

$= \ln|1+e^{2t}| + c$

What to look for:

- u & du for u-sub
- $\frac{1}{u}$ for ln
- $\frac{1}{a^2+u^2}$ for arctan
- $\frac{1}{\sqrt{a^2-u^2}}$ for arcsin

How to: Create a General Formula

General Formulas

As time goes on we encounter more and more complex integrals that require u-substitution within other rules. Your first example of this was probably along the lines of $\int \frac{1}{ax+b}dx$ which is so close to $\int \frac{1}{x}dx = \ln|x| + c$. In fact it is a "simple" u-substitution away (try $u = ax+b$). The u-substitution is often time consuming and uses multiple written steps. We shorten this time through pattern recognition and general formulas.

Let us compute a few of the most commonly used:

(i) $\int \frac{1}{ax+b}\,dx \approx \int \frac{1}{x}dx$

$u = ax+b$

$du = a\,dx \Rightarrow \frac{1}{a}du = dx$

$\int \frac{1}{u}\cdot\frac{1}{a}du$

$= \frac{1}{a}\ln|u| + c$

$= \frac{1}{a}\ln|ax+b| + c$

$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$

(ii) $\int \frac{1}{a^2+b^2x^2}\,dx \approx \int \frac{1}{1+x^2}dx$

$\int \frac{1}{a^2(1+\frac{b^2}{a^2}x^2)}dx$

$\int \frac{1}{a^2(1+(\frac{b}{a}x)^2)}dx$

$u = \frac{b}{a}x$

$du = \frac{b}{a}dx \Rightarrow \frac{a}{b}du = dx$

$\int \frac{1}{a^2(1+u^2)}\cdot\frac{a}{b}du$

$= \frac{1}{ab}\arctan(u) + c$

$= \frac{1}{ab}\arctan(\frac{b}{a}x) + c$

$\int \frac{1}{a^2+b^2x^2}dx = \frac{1}{ab}\arctan(\frac{b}{a}x) + c$

Now you try:

(iii) $\int e^{ax+b}dx \approx \int e^x dx$

(iv) $\int \frac{1}{\sqrt{a^2+b^2x^2}}dx \approx \int \frac{1}{\sqrt{1+x^2}}dx$

Exit Ticket Integral Review

Solve the following integrals and identify the integral rule used:

1. $\int \cot(x)\sin(x)dx$

$= \int \frac{\cos(x)}{\sin(x)} \cdot \sin(x)dx$

$= \int \cos(x)dx$

$= \sin(x) + C$

2. $\int \frac{1+\cos^2(\theta)}{\cos^2(\theta)} d\theta$

$= \int \frac{1}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} d\theta$

$= \int \sec^2\theta + 1 \, d\theta$

$= \tan(\theta) + \theta + C$

3. $\int \frac{\sin(x)}{1+\cos^2(x)} dx$

$u = \cos(x) \quad du = -\sin(x)dx$

$\int \frac{-1}{1+u^2} du$

$= -\arctan(u) + C$

$= -\arctan(\cos(x)) + C$

4. $\int 6x(x^2+1)^{\frac{1}{2}} dx$

$u = x^2+1 \quad du = 2x\,dx$

$\int 3\sqrt{u}\,du$

$= 3 \cdot \frac{2}{3} u^{3/2} + C$

$= 2(x^2+1)^{3/2} + C$

5. $\int \sin(2x)dx$

$= -\frac{1}{2}\cos(2x) + C$

6. $\int \frac{1}{1+\sin(\theta)} d\theta$

$= \int \frac{1}{1+\sin(\theta)} \cdot \frac{1-\sin(\theta)}{1-\sin(\theta)} d\theta$

$= \int \frac{1-\sin(\theta)}{1-\sin^2(\theta)} d\theta$

$= \int \frac{1-\sin(\theta)}{\cos^2(\theta)} d\theta$

$= \int \frac{1}{\cos^2(\theta)} - \frac{\sin(\theta)}{\cos^2(\theta)} d\theta$

$= \int \sec^2(\theta) - \frac{\sin(\theta)}{\cos(\theta)} \cdot \frac{1}{\cos(\theta)} d\theta$

$= \int \sec^2(\theta) - \tan(\theta) \cdot \sec(\theta) d\theta$

$= \tan(\theta) - \sec(\theta) + C$

7. $\int \frac{3x}{(2x^2+1)^2} dx$

$u = 2x^2+1 \quad du = 4x\,dx$

$= \int 3 \cdot \frac{1}{u^2} \cdot \frac{1}{4} du$

$= \frac{3}{4} \cdot \frac{1}{-1} u^{-1} + C$

$= -\frac{3}{4}(2x^2+1)^{-1} + C$

8. $\int \frac{3x}{(2x^2+1)} dx$

$u = 2x^2+1 \quad du = 4x\,dx$

$= \int 3 \cdot \frac{1}{u} \cdot \frac{1}{4} du$

$= \frac{3}{4} \ln|u| + C$

$= \frac{3}{4} \ln|2x^2+1| + C$