Applications of Exponential and Logarithmic Functions

Exponential Growth and Decay

A quantity y is said to grow or decay exponentially if the rate of change of y is proportional to the quantity of y. In other words, y(t) satisfies the differential function:

$$\frac{dy}{dt} = r \cdot y$$

Some real life examples:

(i) exponential growth: bacteria culture, compound intrest, viral spread, etc.

(ii) exponential decay: radioactive decay, drug concentration, depreciation of assets, ...

We call k the rate of growth.

↳ if $r > 0$ then we say that y is growing exponentially

↳ if $r < 0$ then we say that y is decaying exponentially

We can also solve the differential equation such that $y = C \cdot e^{rt}$ where C is the initial value.

Doubling Time and Half Life

If y(t) is exponentially growing, then the time it takes for the initial amount to double is called the doubling time, i.e. the time t such that $2C = C \cdot e^{rt}$.

If y(t) is exponentially decaying, then the time it takes for the initial amount to be cut in half is called the half life, i.e. the time t such that $\frac{1}{2}C = C \cdot e^{rt}$

Examples

1. Recent experiments on viability of the coronavirus indicates that it reduces exponentially on various surfaces. The half life of the coronavirus on glass is estimated to be about 14 hours. (a) Starting with 100% initially, find a formula in the form $A \cdot e^{rt}$ for the percentage of the virus on glass after t hours. (b) If we consider the virus no longer infectious (or viable) after it is reduced to 1% or less, estimate how long will the virus remain infectious on glass.

$A = 100, r = -\frac{\ln(2)}{14}$

Reference:
Aerosol and Surface Stability of SARS-CoV-2 as Compared with SARS-CoV-1, N Engl J Med April 2020

Stability of SARS-CoV-2 in different environmental conditions, Lancet April 2020.

(a) given: half life = 14 hrs, initial = 100% i.e. (0,100) & (14,50)

asked for: $y(t) = A \cdot e^{rt}$ ⟵ 3 unknowns A, r, t

Use initial to solve for initial value A:

$y(0) = A \cdot e^{r \cdot 0}$

$100 = A \cdot 1$

Use second point to solve for rate r:

$y(14) = 100e^{r \cdot 14}$ ⟵ use the A from above

$50 = 100e^{14r}$

$\frac{1}{2} = e^{14r}$ ← left always becomes $\frac{1}{2}$ hence half-life

$\ln(\frac{1}{2}) = \ln(e^{14r})$

$\ln(2^{-1}) = 14r \ln(e)$

$-\frac{1}{14} \cdot \ln(2) = r$

$y(t) = 100e^{-\frac{1}{14}\ln(2)\cdot t}$ ← negative rate = decay

(b) given: no longer infectious $\leq 1\%$ i.e. $y(t) \leq 1$

asked for: time where it is no longer infectious

solve for t: $y(t) = 1$

$1 = 100e^{-\frac{1}{14}\ln(2)\cdot t}$

$\frac{1}{100} = e^{-\frac{1}{14}\ln(2)\cdot t}$

$\ln(\frac{1}{100}) = \ln(e^{-\frac{1}{14}\ln(2)\cdot t})$

$\ln(100^{-1}) = -\frac{1}{14}\ln(2)\cdot t$

$-\ln(100) = -\frac{1}{14}\ln(2)\cdot t$

$14\frac{\ln(100)}{\ln(2)} = t$ ← this does not simplify no matter how much you want it to

≈ 3.88 days

2. A cypress beam found in the tomb of Sneferu in Egypt contained 55% of the amount of Carbon-14 found in living cypress wood. Estimate the age of the tomb given that Carbon-14 has a half-life of 5730 years.

given: tomb wood is at 55%, cypress half life is 5730

asked for: age of tomb wood i.e. T such that $y(T) = 55$

First, we need a formula $y(t) = Ae^{rt}$

$\frac{1}{2} = e^{r\cdot 5730}$

$\ln(\frac{1}{2}) = \ln(e^{5730r})$

$\ln(2^{-1}) = 5730r$

$-\frac{1}{5730}\ln(2) = r$

$y(t) = 100e^{-\frac{1}{5730}\ln(2)\cdot t}$

Solve for T: $y(T) = 55$

$55 = 100e^{-\frac{1}{5730}\ln(2)\cdot T}$

$\frac{55}{100} = e^{-\frac{1}{5730}\ln(2)\cdot T}$

$\ln(\frac{55}{100}) = \ln(e^{-\frac{1}{5730}\ln(2)\cdot T})$

$\ln(\frac{55}{100}) = -\frac{1}{5730}\ln(2)\cdot T$

$-5730\frac{\ln(55/100)}{\ln(2)} = T$ $\quad \frac{a}{b} = (\frac{b}{a})^{-1}$

$-5730\cdot\frac{-\ln(100/55)}{\ln(2)} = T$

$5730\cdot\frac{\ln(100/55)}{\ln(2)} = T$

≈ 4942.105 yrs