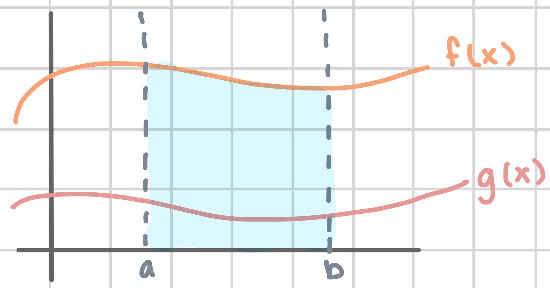
Week 03: Febuary 27th, 2023

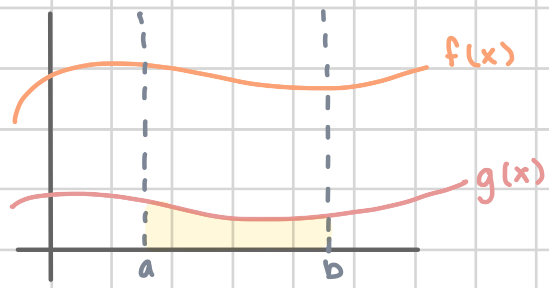
Section 6.1: Area Between Curve

Assuming that $f(x) > g(x)$ for $a < x < b$, find the area between the curves $y = f(x)$ and $y = g(x)$.

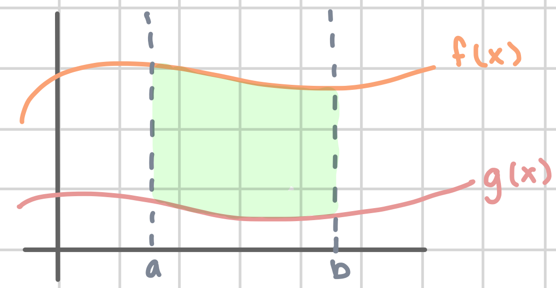
① Breaking down the picture



Area between the graph of f(x) and the x-axis is given by $\int_a^b f(x)\,dx$

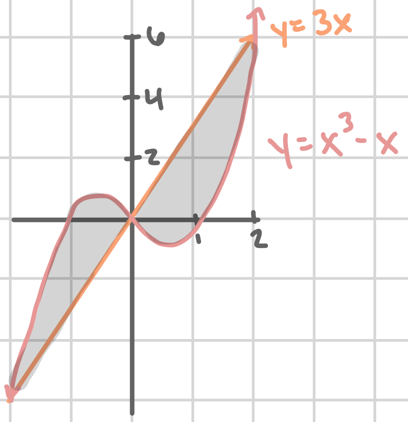


Area between the graph of g(x) and the x-axis is given by $\int_a^b g(x)\,dx$



Area between the graph of f(x) and the graph of g(x) is given by $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

example. Find the area enclosed by the graphs of $y = x^3 - x$ and $y = 3x$.



Area between the curves $= \int_a^b (f(x) - g(x))\,dx$ where f(x) is the "top" function and g(x) is the "bottom" function.
But! Here they switch "top" and "bottom" so we must split our integral where they swap positions $x = 0$.

$\int_{-2}^{0} (x^3 - x) - (3x)\,dx + \int_0^2 (3x) - (x^3 - x)\,dx$

$= \int_{-2}^{0} x^3 - 4x\,dx + \int_0^2 -x^3 + 4x\,dx$

$= \frac{1}{4}x^4 - 2x^2 \Big]_{-2}^{0} + -\frac{1}{4}x^4 + 2x^2 \Big]_0^2$

$= \left[\frac{1}{4}(0)^4 - 2(0)^2 - \left(\frac{1}{4}(-2)^4 - 2(-2)^2\right)\right] + \left[-\frac{1}{4}(2)^4 + 2(2)^2 - \left(-\frac{1}{4}(0)^4 + 2(0)^2\right)\right]$

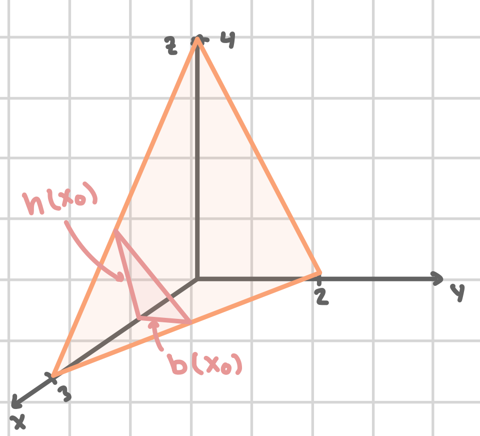
$= 0 - 0 - 4 + 8 - 4 + 8$

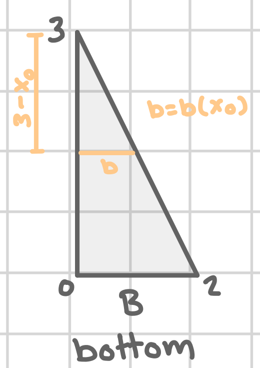
$= 8$

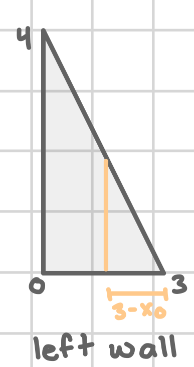
Section 6.2: Volume of a Solid with Uniform Cross-section

You can use the definite integral to find the volume of a solid with specific cross sections on an interval, provided you know a formula for the region determined by each cross section. If the cross sections generated are perpendicular to the x-axis, then their areas will be functions of x, denoted by A(x). The volume (v) of the solid on the interval [a,b] is $V = \int_a^b A(x)\,dx$. Similarly, if the cross sections are perpendicular to the y-axis, then their areas will be functions of y, denoted by A(y). In this case, the volume (v) of the solid [a,b] is $V = \int_a^b A(y)\,dy$.

example. Find the volume of the solid shown below by integrating the area of vertical cross-section perpendicular to the x-axis.







Area of cross section $(\Delta) = \frac{b(x)\,h(x)}{2}$

By similar Δ's: $\frac{2}{3} = \frac{b(x_0)}{3 - x_0}$

$b(x_0) = \frac{2}{3}(3 - x_0)$

Similarly,: $\frac{h(x_0)}{4} = \frac{3 - x_0}{3}$

$h(x_0) = \frac{4}{3}(3 - x_0)$

Area $(\Delta) = \frac{b(x) \cdot h(x)}{2} = \frac{\frac{2}{3}(3-x) \cdot \frac{4}{3}(3-x)}{2}$

$= \frac{8/9\,(3-x)^2}{2}$

$= \frac{4}{9}(3-x)^2$

$V = \int_0^3 A(x)\,dx$

$= \int_0^3 \frac{4}{9}(3-x)^2\,dx$

$= \int_0^3 \frac{4}{9}(9 - 6x + x^2)\,dx$

$= \frac{4}{9}\left[9x - 3x^2 + \frac{1}{3}x^3\right]_0^3$

$= \frac{4}{9}\left[9(3) - 3(3)^2 + \frac{1}{3}(3)^3 - (0 - 0 + 0)\right]$

$= \frac{4}{9}[27 - 27 + 9]$

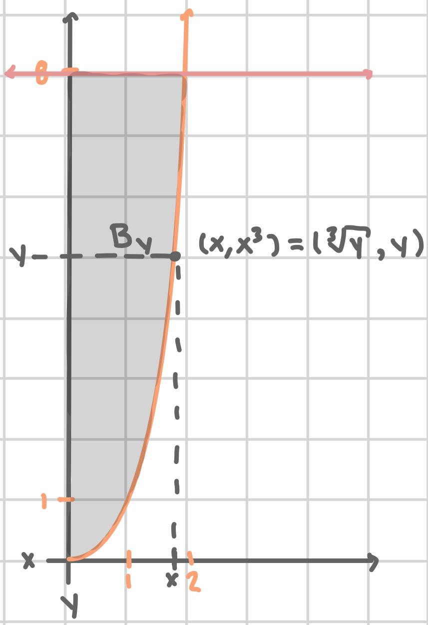
$= \frac{4}{9}[9]$

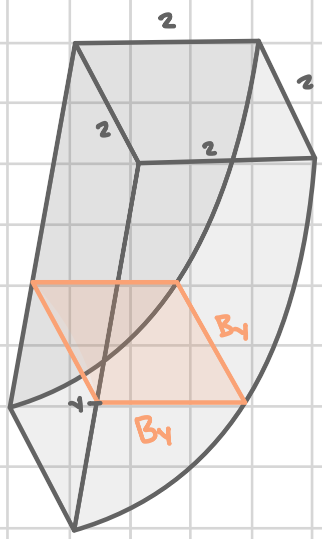
$= 4$

example. Consider a solid whose base is the region bounded by the lines $y=x^3$, $y=8$, and the y-axis. Find the volume of the solid in each of the following cases:

a. The cross sections perpendicular to the y-axis are squares.
b. The cross sections perpendicular to the y-axis are rectangles of height $\sqrt{y}$.
c. The cross sections perpendicular to the y-axis are semicircles.

a. The cross sections perpendicular to the y-axis are squares.



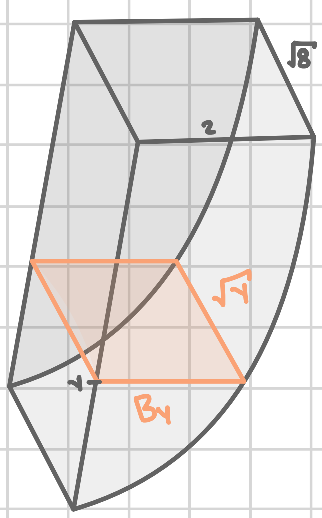


$A(y)$ = Area of cross-section at y
= Area of square with side B_y
$= (B_y)^2$
$= (\sqrt[3]{y})^2$
$= y^{2/3}$

B_y = Base of cross-section when cutting at y

$V = \int_0^8 y^{2/3}\,dy$
$= \left[\frac{1}{2/3+1} y^{2/3+1}\right]_0^8$
$= \left[\frac{1}{5/3} y^{5/3}\right]_0^8$
$= \left[\frac{3}{5} y^{5/3}\right]_0^8$
$= \frac{3}{5}\left((8)^{5/3} - (0)^{5/3}\right)$
$= \frac{3}{5}(8^{1/3})^5$
$= \frac{3}{5}(2)^5$
$= \frac{3}{5}(32)$
$= \frac{96}{5}$

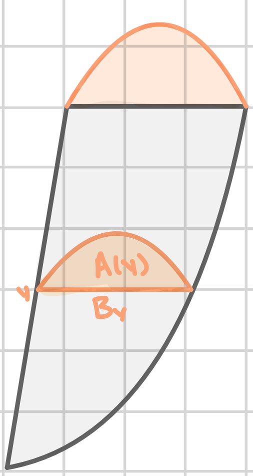
b. The cross sections perpendicular to the y-axis are rectangles of height $\sqrt{y}$.



$A(y)$ = Area of cross-section at y
= Area of rectangle with base $\sqrt[3]{y}$ and height $\sqrt{y}$
$= \sqrt[3]{y}\,\sqrt{y}$
$= y^{1/3} y^{1/2}$
$= y^{5/6}$

$\int_0^8 y^{5/6}\,dy$
$= \left[\frac{1}{5/6+1} y^{5/6+1}\right]_0^8$
$= \left[\frac{1}{11/6} y^{11/6}\right]_0^8$
$= \left[\frac{6}{11} y^{11/6}\right]_0^8$
$= \frac{6}{11}\left(8^{11/6} - 0^{11/6}\right)$
$= \frac{6}{11}(8)^{11/6}$

c. The cross sections perpendicular to the y-axis are semicircles.



Area(y) = Area of cross-section at y
= Area of semicircle of diameter B_x
= Area of semicircle of diameter $\sqrt[3]{y}$
$= \frac{1}{2}\left(\frac{1}{4}\pi(\sqrt[3]{y})^2\right)$
$= \frac{1}{8}\pi y^{2/3}$

$V = \int_0^8 \frac{1}{8}\pi y^{2/3}\,dy$
$= \frac{1}{8}\pi\left(\frac{1}{2/3+1} y^{2/3+1}\right)\Big]_0^8$
$= \frac{1}{8}\pi\left(\frac{1}{5/3} y^{5/3}\right)\Big]_0^8$
$= \frac{1}{8}\pi\left(\frac{3}{5} y^{5/3}\right)\Big]_0^8$
$= \frac{3}{40}\pi\left(8^{5/3} - 0^{5/3}\right)$
$= \frac{3}{40}\pi(8)^{5/}$

Section 6.2: Density

Every continuous radom variable, X, has a probability density function, p(x). Probability density functions satisfy:
(i) $p(x) \geq 0$ for all x
(ii) $\int_{-\infty}^{\infty} p(x)\,dx = 1$

Probability density functions can be used to determine the probability that a continuous random variable lies between two values, say a and b. This probability is denoted by $P(a \leq X \leq b)$ and is given by, $P(a \leq X \leq b) = \int_a^b p(x)\,dx$.

example. Find the total mass of a 5 meter rod whose linear density is given by $\rho(x) = \frac{e^x}{(1+e^x)^2}$ g/m for $0 \le x \le 5$

$$\text{mass} = \int_0^5 \rho(x)\,dx = \int_0^5 \frac{e^x}{(1+e^x)^2}\,dx$$

$u = 1+e^x \quad du = e^x\,dx$

$$= \int_2^{1+e^5} \frac{1}{u^2}\,du$$

$$= \int_2^{1+e^5} u^{-2}\,du$$

$$= -u^{-1}\Big]_2^{1+e^5}$$

$$= -[(1+e^5)^{-1} - (2)^{-1}]$$

$$= \frac{-1}{1+e^5} + \frac{1}{2}$$

example. A variety of deep sea worm is distributed about a hydrothermal vent according to the population density $\rho(r) = \frac{800}{9+r^2}$ thousand per square miles where $1 \le r \le 3$ is the distance (in miles) from the tent. Find the total population of the sea worm.

Total worms on the circle of radius r = $\rho(r)\cdot$ perimeter of the circle = $\rho\cdot(r)\cdot$ circumference of circle of radius r.

Total population = $\int_1^3$ (worms in circle of radius r) dr

$$= \int_1^3 \rho(r)\cdot 2\pi r\,dr$$

$$= \int_1^3 2\pi r\left(\frac{800}{9+r^2}\right)dr$$

$u = 9+r^2 \quad du = 2r\,dr$

$$= \int_{10}^{18} 800\pi\left(\frac{1}{u}\right)du$$

$$= 800\pi \ln|u|\,du$$

$$= 800\pi(\ln|18| - \ln|10|)$$

$$= 800\pi\cdot\ln\left|\frac{9}{5}\right|$$

6.2: Average of a Function

The average of a continuous function $f(x)$ over the interval $[a,b]$ is given by, $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

example. Find the average amount of money over the first 10 years in an account earing interest at an annual rate of 4% compounded continuously if the principle is \$5000. Draw a graph of the balance in the account and mark the value that represents the average amount of money. Find the time it takes the account to reach this average.

Step 1: Set up average

$f_{avg} = \frac{1}{b-a}\int_a^b A(t)\,dt$

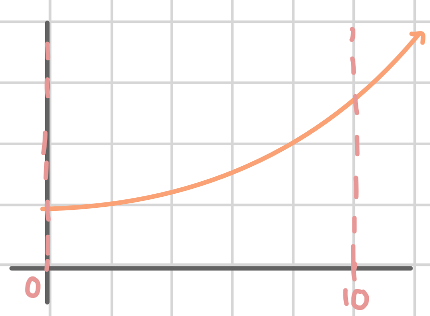
$A(t) = P_0 e^{rt}$

$P_0 = 5000$

$r = 0.04$

$f_{avg} = \frac{1}{10}\int_0^{10} 5000e^{0.04t}\,dt$

Step 2: Draw a graph



Step 3: Solve average

$$f_{avg} = \frac{1}{10}\int_0^{10} 5000\,e^{0.04t}\,dt$$

$$= 500\int_0^{10} e^{0.04t}\,dt$$

$$= 500\cdot\left[e^{0.04t}\cdot\frac{1}{0.04}\right]_0^{10}$$

$$= 12500\left[e^{0.04t}\right]_0^{10}$$

$$= 12500\left(e^{0.04\cdot 10} - e^{0.04\cdot 0}\right)$$

$$= 12500\left(e^{0.4} - 1\right)$$

Step 4: Solve for T such that $A(T) = f_{avg}$

$$5000\,e^{0.04T} = 12500\,(e^{0.4} - 1)$$

$$e^{0.04T} = 2.5\,(e^{0.04} - 1)$$

$$\ln(e^{0.04T}) = \ln(2.5\,(e^{0.04} - 1))$$

$$0.04T = \ln(2.5(e^{0.04} - 1))$$

$$T = \frac{1}{0.04}\cdot\ln(2.5(e^{0.04} - 1))$$